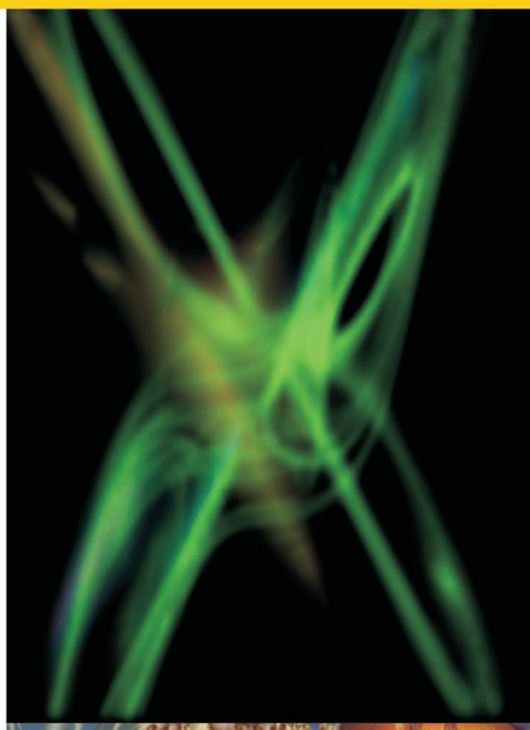
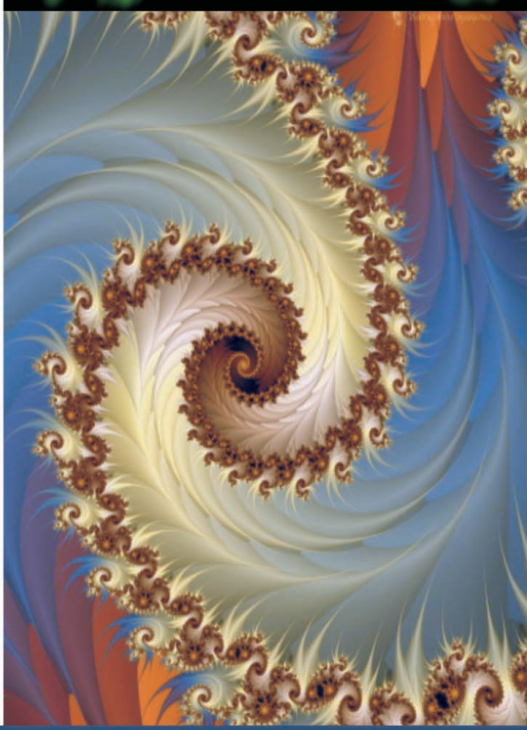
2nd Edition

PRINCIPLES OF REAL ANALYSIS





S C MALIK

New Academic Science

PRINCIPLES OF REAL ANALYSIS

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PRINCIPLES OF REAL ANALYSIS
(SECOND EDITION)

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Preface

The aim has been to provide a development of the subject-matter which is honest, rigorous and at the same time not too pedantic. Most of the hard theorems which are either omitted or treated rather skimpily in many texts in advanced analysis are proved with care. Some of them are ordinarily considered too difficult for a standard course in calculus but too elementary for a course of analysis. With the inclusion of these theorems the book attempts to fill the gap and to make the transfer from elementary calculus to advanced course in analysis as smooth as possible.

The book starts with a quick review of the essential properties of rational numbers. Using Dedekind's Cut the properties of real numbers are established. This foundation supports the subsequent chapters: topological framework, sequences and series of numbers, continuity, differentiation, elementary functions, Riemann integration, with a quick look at the Riemann-Stieltjes integral and finally the functions of several variables and the implicit functions. The material on functions of two variables is given with more details and with more examples and motivation than is usually done. Cantor's theory of the construction of real numbers appears in an appendix to the book which may be studied and enjoyed whenever time seems ripe. Because of the narrow compass of the book, many important topics had to be omitted which could perhaps be presented in a second volume, *Mathematical Analysis* for advanced studies.

A large number of examples, taken mostly from the question papers of different universities and from his own class notes, have been graded and supplied with answers. The majority of them are straight forward; hints for harder ones are occasionally given. Examples to illustrate the general theory or to show where it breaks down follow every important principle. This and the remarks and notes added here and there should help in fixing the ideas better. Effort has also been made to give the proofs of a numbers of theorems in a somewhat modified form.

During the study of the subject and the preparation of lecture notes for more than three decades of my university teaching (where from the present book has taken shape) I have made use of the standard works of a large number of authors including Bromwich, Carslaw, Courant, Ferrar, Gibson, Hyslop, Knopp, Landau and Phillips. I am fully aware that I have benefited largely from them but I cannot resolve, how much is due to each. I owe much to them and I gladly take this opportunity to acknowledging my indebtedness to them.

Finally I may add that the author as also the publishers will welcome all suggestions for improvement of the book.

S.C. Malik

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CHAPTER 1

Real Numbers

1.1 INTRODUCTION

In school algebra and arithmetic we usually deal with two fundamental operations *viz*. *addition* and *multiplication* and their inverse operations, *subtraction* and *division* respectively. These operations are related to a certain class of 'numbers' which will be described more precisely in the following sections. The basic difference between 'elementary mathematics' and 'higher mathematics', which begins at the college level, consists in introducing the all important notion of **limit** which is very intimately related to the intuitive idea of *nearness* or *closeness* and which cannot be described in terms of the operations of addition and multiplication. The notion of limit comes into play in situations where one quantity depends on another varying quantity and we have to know the behaviour of the first when the second is *arbitrarily close* to a fixed given value. In order to illustrate our point in relation to a practical situation, consider the question of determining the velocity of the planet earth at a particular instant during its motion round the sun assuming that the path of its motion round the sun and its position on this path at any instant are known. We cannot determine the velocity of earth without taking recourse to the notion of limit and indeed we need the notion of limit even in defining the concept of 'velocity' of a moving object which is not moving with a uniform speed. The purpose of this illustration is simply to indicate that there are numerous situations where the methods of elementary algebra prove quite inadequate for the purpose of solving or even formulating a problem, and we are forced to evolve new concepts and methods. The notion of *limit* is one such concept.

For the proper understanding of the notion of limit and its importance, it is absolutely desirable that the reader be familiar with the true nature and important properties of 'real numbers'. Starting with natural numbers, we shall briefly and intuitively introduce in the following sections the concept of *rational numbers* and *irrational numbers* which together form the system of *real numbers* describing in the process the important properties possessed by these numbers. The branch of mathematics called *real analysis* deals with problems which are closely connected with the notion of 'limit' and some other notions, such as the operations of 'differentiation' and 'integration' which are directly dependent on the concept of limit when all these operations are confined to the domain of real numbers. It is difficult to say anything more precise at this stage. The interested reader will certainly have a clearer and precise understanding of this important branch of mathematics as he systematically studies this work.

1.1.1 Real Numbers

The system of real numbers has evolved as a result of a process of successive extensions of the system of **natural numbers** (*i.e*., the positive whole numbers). It may be remarked here that the extension became absolutely inevitable as the science of Mathematics developed in the process of solving problems from

other fields. Natural numbers came into existence when man first learnt counting which can also be viewed as adding successively the number 1 to unity. If we add two natural numbers, we get a natural number—but the inverse operation of subtraction is not always possible if we limit ourselves to the domain of natural numbers only. For instance, there is no natural number which added to 8 will give us 3. In other words, 8 cannot be subtracted from 3 within the system of natural numbers. In order that the operation of subtraction (*i.e.*, operation inverse of addition) be also performed without any restriction, it became necessary to enlarge the system of natural numbers by introducing the *negative integers* and the number *zero*. Thus to every natural number n corresponds a unique negative integer designated $-n$ and called the additive inverse of n, and there is a number zero, written 0, such that $n + (-n) = 0$, and $n + 0 = n$ for every natural number n. Also n is the inverse of $-n$. The negative integers, the number 0, and the natural numbers (*i.e.*, the positive integers) together constitute what is known as the system of **integers**. Similarly, to make division always possible, zero being an exception, the concept of fractions, positive and negative, was introduced. Division by zero, however, cannot be defined in a meaningful and consistent manner. The system so extended, including integers and fractions both positive and negative, and the number zero, is called the system of **rational numbers**. Thus, every rational number can be represented in the form p/q where p and q are integers and $q \neq 0$.

We know that the result of performing any one of the four operations of arithmetic (division by zero being, of course, excluded) in respect of any two rational numbers, is again a rational number. So long as mathematics was concerned with these four operations only, the system of rational numbers was sufficient for all purposes but the process of extracting roots of numbers (*e.g.*, square-root of 2, cube-root of 7, etc.), as also the desirability of giving a meaning to non-terminating and non-recurring decimals, necessitated a further extension of the number system. There were lengths which could not be measured in terms of rational numbers, for instance the length of the diagonal of a square whose sides are of unit length, cannot be measured in terms of rational numbers. In fact, this is equivalent to saying that there is no rational number whose square is equal to 2. In order to be able to answer such questions, the system of rational numbers had to be further enlarged by introducing the so called irrational numbers. It is beyond the scope of this book to discuss systematically the definition of irrational numbers in terms of rational numbers. Numbers like $\sqrt{2}, \sqrt[3]{7}, \pi$ (*i.e.*, the ratio of the circumference to the diameter of a circle) with which the reader is already familiar are examples of irrational numbers. Rational numbers and irrational numbers together constitute what is known as the system of **real numbers**.

Though the real number system cannot be extended in a way in which a rational number system is extended but it can be used to develop another system, called the system of *complex numbers*. But since real analysis is not concerned with complex numbers, we shall have nothing to do with complex numbers in this book.

For the sake of brevity and clarity of exposition, and because the notion of set is fundamental to all branches of mathematics, we start with the algebra of sets.

1.1.2 Sets

A set is a well defined collection of objects. In other words, an aggregate or class of objects having a specified property in common which enables us to tell whether any given object belongs to it or not. The individual objects of the set are called *members* or *elements* of the set. Capital letters A, B, C, etc., are generally used to denote the sets while small letters a, b, c, etc., for *elements*. If x is a member of a set A, then we write

$$x \in A$$

and read it as 'x belongs to A' or 'x is an element of A' or 'x is a member of A' or simply 'x is in A'. If x is not a member of A, then we write

$$x \notin A$$

and read it as 'x does not belong to A'.

Some Typical Sets

N: the set of natural numbers,
I: the set of integers,
$\mathbf{I}^+$: the set of positive integers,
$\mathbf{I}^-$: the set of negative integers,
Q: the set of rational numbers,
R: the set of real numbers.

There are two methods which are in common use to denote a set.

(i) A set may be described by listing all its elements.

(a) Set S has elements a, b, c, then we write

$S = \{a, b, c\}$

(b) Set V of vowels in the English alphabets

$V = \{a, e, i, o, u\}$

(ii) A set may be described by means of a property which is common to all its elements.

(a) The set S of all elements x which have the property $P(x)$

$S = \{x: P(x)\}$

(b) The set B of natural numbers

$B = \{n : n \in \mathbf{N}\}$

A Null Set. A set having no element. Sometimes the defining property of a set is such that no object can satisfy it, so that the set remains empty. Such a set is called a **null set**, an **empty set** or a **void set**, and is generally denoted by the Danish letter ϕ or $\{\ \}$.

Thus $\phi = \{n : n$ is a natural number less than $1\}$

$\phi = \{x : x \neq x\}$

1.1.3 Equality of Sets

Two sets are said to be equal when they consist of exactly the same elements. Thus, sets P, Q are equal ($P = Q$) if every element of P is an element of Q and every element of Q is also an element of P. Thus,

$$\{a, b, c\} = \{b, a, c\}$$

$$\{4, 5, 6, 7, 8, 9\} = \{n : 3 < n < 10, n \in N\}$$

It is to be noted that in writing a set, an element occurs only once but the order in which the elements of a set are written is immaterial.

A set is **finite** or **infinite** according to as the number of element in it is finite or infinite.

1.1.4 Notation

$$\forall, \exists, \Rightarrow, \Leftrightarrow, \wedge, \vee, \sim$$

These symbols borrowed from mathematical logic help in a neat and brief exposition of the subject and so we shall describe them here briefly.

(*i*) $\forall$ stands for 'for all' or 'for every'.

The statement $x < y$, $\forall x \in S$ means, x is less than y for all members of S,

i.e., all members of S are less than y.

(*ii*) $\exists$ stands for 'there exists'.

(*iii*) $\Rightarrow$ stands for 'implies that'.

If P and Q are two statements then $P \Rightarrow Q$ means that the statement P implies the statement Q

i.e., if P is true, then Q is also true. Thus

$$x = 5 \Rightarrow x^2 = 25$$

$$AB \| CD \text{ and } CD \| EF \Rightarrow AB \| EF$$

If the statements P and Q are such that P implies Q and Q implies P, then we write

$$P \Leftrightarrow Q \quad \text{(both ways implication)}$$

Thus for real numbers x, y

$$xy = 0 \Leftrightarrow x = 0 \text{ or } y = 0$$

(*iv*) $\wedge$ stands for 'and'

$\vee$ stands for 'or'

The statement $P \wedge Q$ holds when both the statements P and Q hold, but the statement $R \vee S$ can hold when either R holds or S holds, *i.e.*, $R \vee S$ holds when at least one of R and S holds. Thus

$$(x-3)(x-5) < 0 \Rightarrow x > 3 \wedge x < 5$$

$$x^2 = 1 \Rightarrow x = 1 \vee x = -1$$

(*v*) Negation ~ stands for 'not'.

If P is a statement then $\sim P$ is negation of P.

In other words, $\sim P$ denotes 'not P'.

Thus when P holds, $\sim P$ cannot hold and *vice versa*.

$$P \wedge \sim P \text{ is always false,}$$

but

$$P \vee \sim P \text{ is always true.}$$

1.1.5 Subsets

If A and B are two sets such that each member of A is also a member of B *i.e.*, $x \in A \Rightarrow x \in B$, then A is called a **subset** of B (or is contained in B) and we write

$$A \subset B.$$

This is sometimes expressed by saying that B is a **Superset** of A (or contains A) and we write

$$B \supset A.$$

Thus, if A is a subset of B then there is no element in A which is not in B, *i.e.*, $y \notin B \Rightarrow y \notin A$. Consequently, the null set ϕ is a subset of every set, and $A \subset A$ for every set A.

Thus, if $A \subset B$ and $B \subset A$, we write $A = B$.

$A \subseteq B$ allows for the possibility that A and B might be equal. If A is a subset of B and is not equal to B, we say that A is a **proper subset** of B (or is properly contained in B). Thus A is a proper subset of B if every member of A is a member of B and there exists at least one member of B which is not a member of A.

Two sets A and B are said to be **comparable** if either $A \supset B$ or $A \subset B$, otherwise they are not comparable.

1.1.6 Union and Intersection of Sets

Union. If A and B are two sets, then the set consisting of all those elements which belong to A or to B or to both, is called the union of A and B and is denoted by $A \cup B$.

Clearly

$$A \cup \phi = A,$$
$$A \cup A = A$$

and

$$A \cup B = B \cup A.$$

Intersection. If A and B are two sets then, the set consisting of all those elements which belong to both A and B, is called the intersection of A and B and is denoted by $A \cap B$.

Clearly

$$A \cap \phi = \phi,$$
$$A \cap A = A$$
$$A \cap B = B \cap A.$$

Thus, $A \cap B$ consists of elements which are common to A and B.

Two sets A and B are said to be **Disjoint** if they have no common element, *i.e.*, $A \cap B = \phi$.

1.1.7 Union and Intersection of an Arbitrary Family

The operations of forming unions and intersections are primarily binary operations, that is, each is a process which applies to a pair of sets and yields a third. We emphasize this by the use of parentheses to indicate the order in which the operations are to be performed, as in $(A_1 \cup A_2) \cup A_3$, where the parentheses direct us first to unite A_1 and A_2, and then to unite the result with A_3. Associativity makes it possible to dispense with parentheses in an expression like this and to write $A_1 \cup A_2 \cup A_3$, where we understand that these sets are to be united in any order and that the order in which the operations are performed is irrelevant. Similar remarks apply to $A_1 \cap A_2 \cap A_3$. Furthermore, if $\{A_1, A_2, ..., A_n\}$ is any **finite** class of sets, then we can form

$$A_1 \cup A_2 \cup ... \cup A_n \text{ and } A_1 \cap A_2 \cap ... \cap A_n$$

in much the same way without any ambiguity of meaning whatever. In order to shorten the notation, we let $I = \{1, 2, ..., n\}$ be the set of subscripts which index the set under consideration. I is called the **Index Set.** We then compress the symbols and write

$$\bigcup_{i \in I} A_i \text{ and } \bigcap_{i \in I} A_i \quad \text{or} \quad \bigcup_{i=1}^{n} A_i \text{ and } \bigcap_{i=1}^{n} A_i$$

It is often necessary to form unions and intersections of large (really large) class of sets. Let Λ be a set and $\{A_\lambda : \lambda \in \Lambda\}$ an entirely arbitrary class or family $\mathscr{F}$ of sets which contains a set A_λ for each λ in Λ. Then

$$\bigcup_{\lambda \in \Lambda} A_\lambda = \{x : x \in A_\lambda \text{ for at least one } \lambda \text{ in } \Lambda\}$$

$$\bigcap_{\lambda \in \Lambda} A_\lambda = \{x : x \in A_\lambda \text{ for every } \lambda \text{ in } \Lambda\}$$

define the union and intersection of an **arbitrary family** $\mathscr{F}$**.**

Λ is called the **Index set**.

If the class $\{A_i\}$ consists of a **sequence** of sets, $\{A_i\} = \{A_1, A_2, ...\}$ then their union and intersection are often written in the form

$$\bigcup_{i=1}^{\infty} A_i \text{ and } \bigcap_{i=1}^{\infty} A_i.$$

1.1.8 Universal Set

In any discussion of sets, all sets are usually assumed to be subsets of a set, called the **universal set** (usually denoted by $\bigcup$). In our present discussion, however, the set **R** of real numbers can serve as the universal set.

1.1.9 Difference Set, Complement of a Set

If A and B are two sets then the set consisting of those elements of B which do not belong to A is called the **difference set** of A and B and is denoted by $B \sim A$ or $B - A$.

If, however, A is a subset of B then $B - A$ is called the **complement of *A* in *B*** or complement of A with respect to B.

Complement of A in the universal set $\bigcup$ is called the **complement** of A and is denoted by A or $\sim A$.

1.1.10 Compositions

We shall be dealing mainly with number sets and so we define only two types of compositions in the sets.

An **Addition Composition** is defined in a set S if to each pair of members a, b of S there corresponds a member $a + b$ of S.

Similarly, a **Multiplication Composition** is defined in S if to each pair of members a, b of S there corresponds a member ab of S.

A set is said to possess an **algebraic structure** if the two compositions of Addition and Multiplication are defined in the set.

Subtraction and *Division* may be defined as inverse operations of addition and multiplication respectively.

Let $a, b \in S$.

Subtraction: $a - b$ may be expressed as $a + (-b)$ when $-b \in S$.

Division: The quotient $a/b\ (b \neq 0)$ may be put as $a \cdot 1/b$ or ab^{-1} when $1/b$ or $b^{-1} \in S$.

1.2 FIELD STRUCTURE AND ORDER STRUCTURE

1.2.1 Field Structure

A set S is said to be a **field** if two compositions of *Addition* and *Multiplication* defined in it be such that $\forall\ a, b, c \in S$ the following properties are satisfied:

A-1. Set S is closed for addition,

$$a, b \in S \Rightarrow a + b \in S$$

A-2. Addition is commutative,

$$a + b = b + a$$

A-3. Addition is associative,

$$(a + b) + c = a + (b + c)$$

A-4. Additive identity exists, *i.e.*, $\exists$ a member 0 in S such that

$$a + 0 = a$$

A-5. Additive inverse exists, *i.e.*, to each element $a \in S$ there exists an element $-a \in S$ such that

$$a + (-a) = 0$$

M-1. S is closed for multiplication,

$$a, b \in S \Rightarrow ab \in S$$

M-2. Multiplication is commutative,

$$ab = ba$$

M-3. Multiplication is associative,

$$(ab)c = a(bc)$$

M-4. Multiplicative identity exists *i.e.*, $\exists$ a member 1 in S such that

$$a \cdot 1 = a$$

M-5. Multiplicative inverse exists *i.e.*, to each $0 \neq a \in S$, $\exists$ an element $a^{-1} \in S$ such that

$$aa^{-1} = 1$$

AM. Multiplication is distributive with respect to addition, *i.e.*,

$$a(b + c) = ab + ac$$

Thus, a set S has a **field** structure if it possesses the two compositions of addition and multiplication and satisfies the eleven properties listed above.

1.2.2 Order Structure

Ordinarily the order relation does not exist between the members of a general field, but as we are to deal with the field of real numbers, we can speak of one number being 'greater than' (or less than) the other.

A *field S* is an **ordered field** if it satisfies the following properties:

O-1. *Law of Trichotomy*: For any two elements $a, b \in S$, one and only one of the following is true.

$$a > b,\ a = b,\ b > a$$

O-2. *Transitivity*: $\forall\ a, b, c \in S$,

$$a > b \wedge b > c \Rightarrow a > c$$

O-3. *Compatibility* of Order Relation with Addition Composition:

$$\forall\ a, b, c \in S,$$

$$a > b \Rightarrow a + c > b + c$$

O-4. *Compatibility* of Order Relation with Multiplication Composition:

$$\forall\ a, b, c \in S,$$

$$a > b \wedge c > 0 \Rightarrow ac > bc$$

1.2.3 It may be seen that the set **Q** of rational numbers and the set **R** of real numbers are ordered fields while the set **N** of natural numbers and the set **I** of integers are not fields.

(*i*) **The Set N of Natural Numbers**

1, 2, 3, 4, ...

The sum or product of any two members is easily seen to be a member of **N**, so that the set possesses two compositions of addition and multiplication, *i.e*., the set **N** possesses an *algebraic structure*. However, it does not satisfy all the properties of a field (it does not possess additive identity, additive inverse and multiplicative inverse) and hence the set of natural numbers is not a field. However, it has an *order structure* compatible with the *algebraic structure*.

(*ii*) **The Set I of Integers**

..., –3, –2, –1, 0, 1, 2, 3, ...

It may be easily seen that the set possesses an *algebraic structure* but does not satisfy all the properties of a field. (M-5 Multiplicative inverses do not exist.) Hence, the set of integers is not a field. However, it has an *order structure* compatible with the *algebraic structure*.

(*iii*) **The Set Q of Rational Numbers**

A rational number is of the form p/q, where p and q are integers and $q \neq 0$. Evidently, the set **Q** of rational numbers includes the set of integers.

A real number which is not rational (*i.e*., cannot be expressed as p/q) is called an **irrational number**.

The set **R** of **real numbers** consists of rational and irrational numbers.

The sets **Q** and **R** satisfy all the properties (§ 2.1) of a field and are therefore fields. In addition to this, both these fields satisfy the four properties 0–1 to 0–4 (§ 2.2) of order and hence form **ordered fields**.

1.2.4 Up to this stage we have discussed two properties—*the field property* and *the order structure property*. We have found that both the sets, the set **R** of real numbers and the set **Q** of rational numbers possess these properties. However, there is a property called the property of **completeness** which is possessed only by the set of real numbers and this distinguishes it from other sets of numbers. Let us, now, consider some notions and examples which will facilitate the study of that property.

1.2.5 Example 1.1. Show that there is no rational number whose square is 2.

Solution. Let, if possible, there exist a rational number p/q, where $q \neq 0$ and p, q are integers prime to each other (*i.e.*, having no common factor) whose square is equal to 2,

i.e., $$(p/q)^2 = 2 \text{ or } p^2 = 2q^2 \quad ...(1)$$

Now q is an integer and so is $2q^2$. Thus, p^2 is an integer divisible by 2. As such p must be divisible by 2, for otherwise p^2 would not be divisible by 2.

Let $p = 2m$, where m is an integer.

Then, from (1), $$2m^2 = q^2 \quad ...(2)$$

Thus, it follows that q is also divisible by 2. Hence, p and q are both divisible by 2 which contradicts the hypothesis that p and q have no common factor. Thus, there exists no rational number whose square is 2.

Example 1.2. Show that $\sqrt{8}$ is not a rational number.

Solution. Let, if possible, $\sqrt{8}$ be the rational number p/q, where $q \neq 0$ and p, q are positive integers prime to each other, so that

$$\sqrt{8} = p/q$$

But $$2 < \sqrt{8} < 3$$

$\therefore$ $$2 < p/q < 3 \Rightarrow 2q < p < 3q \text{ or } 0 < p - 2q < q$$

Thus, $p - 2q$ is a positive integer less than q, so that

$$\sqrt{8}(p - 2q) \text{ or } p/q\,(p - 2q) \text{ is not an integer.}$$

But $$\sqrt{8}(p - 2q) = p/q(p - 2q) = \frac{p^2}{q} - 2p$$

$$= \frac{p^2}{q^2}q - 2p = 8q - 2p, \text{ which is an integer.}$$

$\Rightarrow$ $$\sqrt{8}(p - 2q) \text{ is an integer.}$$

Thus we arrive at a contradiction.

Hence, $\sqrt{8}$ is not a rational number.

Remark. We have considered $\sqrt{n}$ (n not a perfect square), first when n was a prime and then n as a composite number. The procedures shown are typical and may be adopted under similar situations.

Exercise. Show that there is no rational number whose square is

(*i*) 3 (*ii*) 5 (*iii*) 6.

1.2.6 Intervals—Open and Closed

We shall now define different types of intervals.

Open Interval. If a and b be two *real numbers* such that $a < b$ then the set

$$\{x : a < x < b\}$$

consisting of *all real numbers* between a and b (excluding a and b) is called an open interval and is denoted by $]a, b[$.

Closed Interval. The set

$$\{x : a \leq x \leq b\}$$

consisting of a, b and all real numbers lying between a and b is called a **closed interval** and is denoted by $[a, b]$.

Semi-closed or Semi-open Intervals

The intervals $$]a, b] = \{x : a < x \leq b\} \text{ and,}$$

$$[a, b[= \{x : a \leq x < b\} .$$

are semi-closed or semi-open. The former is open at a and closed at b while the latter is closed at a and open at b.

1.3 BOUNDED AND UNBOUNDED SETS: SUPREMUM, INFIMUM

A set S of real numbers is said to be **bounded above** if $\exists$ a real number $\boldsymbol{K}$ such that every member of $\boldsymbol{S}$ is less than or equal to $\boldsymbol{K}$,

i.e., $$x \leq K, \ \forall \ x \in S.$$

The number $\boldsymbol{K}$ is called an **upper bound** of $\boldsymbol{S}$. If no such number $\boldsymbol{K}$ exists, the set is said to be **unbounded above** or **not bounded above**.

The set S is said to be **bounded below** if $\exists$ a real number k such that every member of S is greater than or equal to k,

i.e., $$k \leq x, \ \forall \ x \in S.$$

The number k is called a **lower bound** of S. If no such number k exists, the set is said to be **unbounded below** or **not bounded below.**

A set is said to be **bounded** if it is bounded above as well as below.

It may be seen that if a set has one upper bound, it has an infinite number of upper bounds. For if K is an upper bound of a set S then every number greater than K is also an upper bound of S. Thus every set S bounded above determines an infinite set—the set of its upper bounds. This set of upper bounds is bounded below in as much as every member of S is a lower bound thereof. Similarly, a set S bounded below determines an infinite set of its lower bounds, which is bounded above by the members of S.

A member G of a set S is called the **greatest** member of S if every member of S is less than or equal to G, *i.e.*,

(*i*) $G \in S$

(*ii*) $x \leq G, \qquad \forall \ x \in S$

Similarly, a member g of the set is its **smallest** (or the least) member if every member of the set is greater than or equal to g.

Clearly, a set may or may not have the greatest or the least member but an upper (lower) bound of the set, if it is a member of the set, is its greatest (least) member. A finite set always has the greatest as well as the smallest member.

If the set of all upper bounds of a set S has the smallest member, say M, then M is called the **least upper bound** (l.u.b.) or the **supremum** of S.

Clearly, the supremum of a set S may or may not exist and in case it exists, it may or may not belong to S. The fact that supremum M is the smallest of all the upper bounds of S may be described by the following two properties:

(i) M is the upper bound of S,

i.e., $$x \leq M, \ \forall \ x \in S.$$

(ii) No number less than M can be an upper bound of S, *i.e.*, for any positive number ε, however small, $\exists$ a number $y \in S$ such that

$$y > M - \varepsilon$$

Again it may be seen that *a set cannot have more than one supremum*. For, let if possible, M and M' be two suprema of a set S, so that M and M' are both upper bounds of S.

Also M is the l.u.b. and M' is an upper bound of S.

$\therefore$ $$M \leq M' \qquad ...(1)$$

Again M' is the l.u.b. and M is an upper bound of S.

$\therefore$ $$M' \leq M \qquad ...(2)$$

From (1) and (2) it follows that $M = M'$.

If the set of all lower bounds of a set S has the greatest member say m then m is called the **Greatest lower bound** (g.l.b.) or the **Infimum** of S.

Like the supremum, the infimum of a set may or may not exist and it may or may not belong to S. It can be easily shown that a set cannot have more than one infimum.

The infimum m of a set S has the following two properties:

(i) m is the lower bound of S,

i.e. $$m \leq x, \ \forall \ x \in S.$$

(ii) No number greater than m can be a lower bound of S, *i.e.*, for any positive number ε, however small, $\exists$ a number $z \in S$ such that

$$z < m + \varepsilon.$$

ILLUSTRATIONS

1. The set **N** of natural numbers is bounded below but not bounded above. 1 is a lower bound.
2. The sets **I**, **Q** and **R** are not bounded.
3. Every finite set of numbers is bounded.
4. The set S_1 of all positive real numbers

   $S_1 = \{x : x > 0, \ x \in \mathbf{R}\}$ is not bounded above, but is bounded below. The infimum 0 is not a member of the set S_1.
5. The infinite set $S_2 = \{x : 0 < x < 1, \ x \in \mathbf{R}\}$ is bounded with supremum 1 and infimum 0, both of which do not belong to S_2.
6. The infinite set $S_3 = \{x : 0 \leq x \leq 1, \ x \in \mathbf{Q}\}$ is bounded, with supremum 1 and infimum 0 both of which are members of S_3.

7. The set $S_4 = \left\{\frac{1}{n} : n \in \mathbf{N}\right\}$ is bounded. The supremum 1 belongs to S_4 while infimum 0 does not.
8. Each of the following intervals is bounded:

$$[a, b],\ ]a, b],\ [a, b[,\ ]a, b[.$$

Example 1.3. Prove that the greatest member of a set, if it exists, is the supremum (l.u.b.) of the set.

Solution. Let G be the greatest member of the set S.

Clearly,

$$x \leq G, \quad \forall\ x \in S$$

so that G is an upper bound of S.

Again no number less than G can be an upper bound of S for if y be any number less than G, there exists at least one member g of S which is greater than y.

Thus, G is the least of all the upper bounds of S, *i.e.*, G is the supremum of S.

EXERCISE

1. Give examples of sets which are
   (*i*) Bounded
   (*ii*) Not bounded
   (*iii*) Bounded below but not bounded above
   (*iv*) Bounded above but not bounded below.
2. Find the infimum and the supremum of the following sets. Which of these belongs to the set?

   (*i*) $[1, 3, 5, 7, 9]$

   (*ii*) $\left\{-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots\right\}$

   (*iii*) $\left\{\frac{1}{n}; n \in \mathbf{N}\right\}$

   (*iv*) $\left\{\frac{(-1)^n}{n}; n \in \mathbf{N}\right\}$

   (*v*) $\left\{-2, -\frac{3}{2}, -\frac{4}{3}, -\frac{5}{4}, \ldots, -\frac{n+1}{n}, \ldots\right\}$

   (*vi*) $\left\{1 + \frac{(-1)^n}{n}; n \in \mathbf{N}\right\}$

   (*vii*) $]a, b[$

   (*viii*) $[a, b[$.
3. Which of the sets in question 2 are bounded?
4. Find the smallest and the greatest members (if they exist) for sets in question 2.
5. Show that the greatest (or the smallest) member of a set, in case it exists, is unique.
6. Show that the smallest member of a set, if it exists, is the infimum of the set.
7. Is the converse of the solved example 1.3, true?

ANSWERS

2. (*i*) 1, 9; both
   (*ii*) $-1, 0$; infimum
   (*iii*) 0, 1; supremum
   (*iv*) $-1, \frac{1}{2}$; both
   (*v*) $-2, -1$; infimum
   (*vi*) $0, \frac{3}{2}$; both
   (*vii*) a, b; none
   (*viii*) a, b; infimum.

3. All sets are bounded.
4. (*i*) 1, 9 (*ii*) –1, does not exist
   (*iii*) does not exist, 1 (*iv*) $-1, \frac{1}{2}$
   (*v*) –2, does not exist (*vi*) $0, \frac{3}{2}$
   (*vii*) do not exist (*viii*) *a*, does not exist.

1.4 COMPLETENESS IN THE SET OF REAL NUMBERS

We have seen that all the properties—the properties of an ordered field, described so far, are possessed by the two sets, the set of real numbers **R** and the set of rational numbers **Q**. We shall now state a property, the property of *completeness* (or *order-completeness*) which is possessed by **R** and not by **Q**. This property not only distinguishes **R** from **Q**, but together with the ordered field property, it characterises **R** *i.e*., the set of real numbers is the only set which is a *Complete Ordered Field*.

1.4.1 Order-Completeness in R

(**O.C.**) *Every non-empty set of real numbers which is bounded above has the supremum* (*or the least upper bound*) *in* **R**.

In other words, the set of upper bounds of a non-empty set of real numbers bounded above has the smallest member.

If *S* is a set of real numbers which is bounded above, then by considering the set $T = \{x : -x \in S\}$ we may state the completeness property in the alternative form as:

Every non-empty set of real numbers which is bounded below has the infimum (or *g.l.b.*) *in* **R**. Or, equivalently the set of lower bounds of a non-empty set of real numbers bounded below has the greatest member.

We have thus completed the description of the set of real numbers as a **Complete Ordered Field**. We shall, however, show that the property of *completeness* does not hold good for the ordered field of rational numbers, *i.e*., the ordered field **Q** of rational is not order complete.

Theorem 1.1. *The set of rational numbers is not order-complete.*

To show that the set of rational numbers does not possess the property of completeness, it will suffice to show that there exists a non-empty set *S* of rationals (a subset of **Q**) which is bounded above but does not have a supremum in **Q**, *i.e*., no rational number exists which can be the supremum of *S*.

Let *S* be the set (a subset of **Q**) of all those positive rational numbers whose square is less than 2, *i.e*.,

$$S = \{x : x \in \mathbf{Q}, x > 0 \wedge x^2 < 2\}$$

Since $1 \in S$, the set *S* is non-empty.

Clearly 2 is an upper bound of *S*, therefore, *S* is bounded above.

Thus, *S* is a non-empty set of rational numbers, bounded above. Let, if possible, the rational number *K* be its least upper bound. Clearly *K* is positive. Also by the law of trichotomy (0 – 1) which holds good in **Q**, one and only one of (*i*) $K^2 < 2$, (*ii*) $K^2 = 2$, (*iii*) $K^2 > 2$ holds.

(*i*) $K^2 < 2$. Let us consider the positive rational number

$$y = \frac{4 + 3K}{3 + 2K}$$

Then

$$K - y = K - \frac{4 + 3K}{3 + 2K} = \frac{2\left(K^2 - 2\right)}{3 + 2K} < 0$$

$$\Rightarrow \qquad y > K \qquad \ldots(1)$$

Also,

$$2 - y^2 = 2 - \left(\frac{4 + 3K}{3 + 2K}\right)^2 = \frac{2 - K^2}{\left(3 + 2K\right)^2} > 0$$

$$\Rightarrow \qquad y^2 < 2 \Rightarrow y \in S \qquad \ldots(2)$$

Thus, the member y of S is greater than K, so that K cannot be an upper bound of S and hence, there is a contradiction.

(*ii*) $K^2 = 2$. We have already shown that there exists no rational number whose square is equal to 2. Thus, this case is not possible.

(*iii*) $K^2 > 2$. Considering the positive rational number y as defined in case (*i*), we may easily deduce from (1) and (2) respectively that

$$y < K \text{ and } y^2 > 2$$

Hence, there exists an upper bound y of S smaller than the least upper bound K, which is a contradiction.

Thus, none of the three possible cases holds. Hence, our supposition that a rational number K is the least upper bound of S is wrong. Thus, no rational number exists which can be the least upper bound of S.

Note. If we admit K in **R** and regard **S** as a set of real numbers then by the order completeness property, the supremum K of S exists in **R**. Clearly $K > 0$ and

$$K^2 < 2 \Rightarrow y^2 < 2 \wedge y > K \Rightarrow K \neq \text{Sup } S$$

$$K^2 > 2 \Rightarrow y^2 > 2 \wedge y < K \Rightarrow K \neq \text{Sup } S$$

Thus by property 0–1, it follows that $K^2 = 2$, *i.e.*, the least upper bound K exists whose square is equal to 2. Further, since $K \notin \mathbf{Q}$, it follows that K is an irrational number. Similarly, it may be seen that there exist real numbers other than rational numbers whose squares are 2, 5, 7, ... etc. This establishes the *existence of irrational numbers*.

Exercise. Show that the set of natural numbers is order-complete.

1.4.2 Archimedean Property of Real Numbers

The order-completeness property has important consequences, one of which is the Archimedean property of real numbers which we now proceed to prove.

Theorem 1.2. *The real number field is Archimedean, i.e., if a and b be any two positive real numbers then there exists a positive integer n such that na > b.*

Let a, b be any two positive real numbers. Let us suppose, if possible, that for all positive integers $n(\in \mathbf{I}^+)$, $na \le b$.

Thus, the set $S = \{na : n \in \mathbf{I}^+\}$ is bounded above, b being an upper bound. By the completeness property of the ordered-field of real numbers, set S must have the supremum M.

$$\therefore \qquad na \le M, \qquad \forall\ n \in \mathbf{I}^+$$

$$\Rightarrow \qquad (n+1)\, a \le M, \qquad \forall\ n \in \mathbf{I}^+$$

$$\Rightarrow \qquad na \le M - a, \qquad \forall\ n \in \mathbf{I}^+$$

i.e., $M - a$ is an upper bound of S.

Thus a number, $M - a$ less than the supremum M (l.u.b.) is an upper bound of S, which is a contradiction and hence our supposition is wrong.

Hence the theorem.

Corollary 1. If a be a positive real number and b any real number then there exists a positive integer n such that $na > b$.

Corollary 2. For any positive real number a there exists a positive integer n such that $n > a$.

The result follows by considering the two positive real numbers 1 and a.

Corollary 3. If a be any real number then there exists a positive integer n such that $n > a$.

For $a \le 0$, any positive integer $n > a$, and for $a > 0$, result follows by cor. 2.

1.4.3 Dedekind's Form of Completeness Property

We now state the completeness property of real numbers in another form, due to Dedekind, which states:

If all the real numbers be divided into two non-empty classes L and U such that every member of L is less than every member of U, then there exists a unique real number, say α, such that every real number less than α belongs to L and every real number greater than α belongs to U.

Clearly, the two classes L and U so defined are disjoint and the number α itself belongs either to L or U. The property of real numbers referred to above is known as *Dedekind's property*. We may restate.

Dedekind's Property. If L and U are two subsets of $\mathbf{R}$ such that

(*i*) $L \neq \phi, U \neq \phi$ (each class has at least one member),

(*ii*) $L \cup U = \mathbf{R}$ (every real number has a class)

(*iii*) Every member of L is less than every member of U,

i.e., $$x \in L \wedge y \in U \Rightarrow x < y$$

Then either L has the greatest member or U has the smallest member.

1.4.4 Let us now prove the **equivalence** of the two forms of completeness.

(*a*) First we show that the *order completeness property of real numbers implies Dedekind's property*. The set $\mathbf{R}$ has the order completeness property *i.e.*, every non-empty subset of $\mathbf{R}$ which is bounded above (below) has the Supremum (Infimum).

Let L, U be two subsets of $\mathbf{R}$ such that

(*i*) $L \neq \phi, U \neq \phi$;

(*ii*) $L \bigcup U = \mathbf{R}$; and

(*iii*) Every member of L is less than every member of U.

We have to show that either L has the greatest member or U has the smallest.

By (*iii*) the non-empty set L is bounded above. If L has the greatest member, it establishes the result. If L has no greatest member, then by the order completeness property, the set of its upper bounds, which coincides with U, has the smallest member. This either L has the greatest member or U has the smallest member.

(*b*) Let, now, $\mathbf{R}$ *satisfy the Dedekind's property*. We shall show that $\mathbf{R}$ also satisfies the order completeness property.

Let S be a non-empty set of real numbers bounded above, then we have to prove that S has the supremum.

Let L and U be two sets of real numbers defined by

$L = \{x: x \text{ is not an upper bound of } S\}$,

$U = \{x: x \text{ is an upper bound of } S\}$.

It may be easily seen that

(*i*) $L \neq \phi, U \neq \phi$,

(*ii*) $L \bigcup U = \mathbf{R}$, and

(*iii*) $x \in L \wedge y \in U \Rightarrow x < y$.

Then by Dedekind property, either L has the greatest member or U has the smallest member.

We shall show that L cannot have the greatest member.

Let, if possible, L have the greatest member, say ξ. Then

$\xi \in L \Rightarrow \xi$ is not an upper bound of S.

$\Rightarrow \exists$ an $a \in S$ such that $\xi < a$.

Now the real number $\dfrac{\xi + a}{2}$ is such that

$$\xi < \frac{\xi + a}{2} < a$$

Since $\dfrac{\xi + a}{2}$ is greater than the greatest member ξ of L,

$$\therefore \quad \frac{\xi + a}{2} \in U$$

$$\Rightarrow \quad \frac{\xi + a}{2} \text{ is an upper bound of } S. \qquad \text{...(1)}$$

Again, since $\dfrac{\xi + a}{2}$ is less than the member a of S,

$$\therefore \qquad \frac{\xi + a}{2} \in L$$

$$\Rightarrow \qquad \frac{\xi + a}{2} \text{ is not an upper bound of } S. \qquad \ldots(2)$$

Thus, we arrive at contradictory conclusions, and as such L has no greatest member. Thus, it follows that U, the set of upper bounds of S, has the smallest member, *i.e.*, the set S has the supremum.

We have thus proved the equivalence of Dedekind's and the order completeness property of **R**.

1.4.5 Explicit Statement of the Properties of the Set of Real Numbers as a Complete-Ordered Field

The set **R** of real numbers is a *complete-ordered field* because for arbitrary members, a, b, c of **R**, it satisfies the following conditions:

A-1. $a, b \in \mathbf{R} \Rightarrow a + b \in \mathbf{R}$

A-2. $a + b = b + a$

A-3. $(a + b) + c = a + (b + c)$

A-4. $\exists$ a member 0 in **R** such that
$a + 0 = a$

A-5. To each $a \in \mathbf{R}$, $\exists$ an element $-a \in R$ such that
$a + (-a) = 0$

M-1. $a, b \in \mathbf{R} \Rightarrow ab \in \mathbf{R}$

M-2. $ab = ba$

M-3. $(ab)c = a(bc)$

M-4. $\exists$ a member 1 in **R** such that
$a \cdot 1 = a$

M-5. To each $a \neq 0 \in \mathbf{R}$, $\exists$ an element $a^{-1} \in \mathbf{R}$ such that
$aa^{-1} = 1$.

AM. $a(b + c) = ab + ac$.

O-1. For any two elements a, b of **R**, one and only one of the following is true:
$a > b, \ a = b, \ b > a$

O-2. $a > b \wedge b > c \Rightarrow a > c$

O-3. $a > b \Rightarrow a + c > b + c$

O-4. $a > b \wedge c > 0 \Rightarrow ac > bc$

O.C. Every non-empty subset of **R** which is bounded above (below) has the supremum (infimum) in **R**.

1.4.6 Representation of Real Numbers as Points on a Straight Line

Points on a line can be used to represent real numbers. This geometrical representation of real numbers is sometimes very useful and suggestive especially to the beginner. But this should not stop us from giving the proper proof of a theorem which may otherwise seem to be obvious.

Let $X'X$ be a straight line. Mark two points O and A on it such that A is to the right of O.

The point O divides the line $X'X$ into two parts; the part to the right of O, containing A, may be called positive and that to the left of O as negative. Such a line for which positive and negative sides are fixed is called a *directed line*.

Let us consider the points O and A to represent rational numbers zero and 1 respectively, so that the distance OA is unity on a certain scale. To represent a rational number m/n ($n > 0$), take a point P on the right of O if m is positive and to the left of O if m is negative, such that OP in m times the nth part of the unit length OA. Of course the point P coincides with O if m is zero. The point P thus represents the rational number m/n. We may say that the rational number m/n corresponds to the point P or the point P corresponds to the rational number m/n. This way any rational number can be made to correspond to a point on the line. If points on the line corresponding to rational numbers be termed as *rational points*. We see that infinite number of rational points lie between any two different rational points, *i.e.*, *between any two rationals, there lie infinitely many rationals*.

Even though the rational points seem to cover a straight line very closely, there remain points on the line which are not rational. For example, the point Q on the line such that OQ is equal to the diagonal of the square with side OA does not correspond to any rational number. Also a point R such that OR is any rational multiple of OQ is also such a point. In fact there are infinitely many such points on the line. Hence, the set **Q** of rational numbers is insufficient to provide a complete picture of the straight line.

Such points on the line which are not rational, and which may be supposed to fill up the gaps between rational points are called *irrational points* and these correspond to irrational numbers. In fact, there is at least one irrational between two rationals. Thus like rationals, there are infinitely many irrationals. Hence, every real number can be represented on the directed line and there seem to be as many points on the directed line as the real numbers. The same fact is expressed by **Dedekind-Cantor Axiom** which states:

To every real number there corresponds a unique point on a directed line and to every point on a directed line there corresponds a unique real number.

In view of the order completeness property, the set of real numbers **R** does not have gaps of the kind **Q** has, and thus forms a continuous system. On account of this characteristic, the set **R** is called the *Arithmetical Continuum* and the set of points on a line as the *Geometrical Continuum*. In view of the above axiom, we see that there is a one-one correspondence between the two continuum and accordingly we may use the word *point* for *real number*, and the *real line* for the *directed line*. It is evident that between any two real numbers, there exist infinitely many real numbers both rational and irrational. This is the property of *denseness* of the real number system.

1.5 ABSOLUTE VALUE OF A REAL NUMBER

The *absolute value*, the *numerical value* or the *modulus* of a real number x, denoted by $|x|$, is defined as

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Thus, we always have

$$|x| \geq 0$$

Also by definition

$$|-x| = |x|$$

Some theorems which are immediate consequences of the definition will now follow:

Theorem 1.3. $|x| = max\,(x, -x)$.

Now $|x| = x \geq -x$, if $x \geq 0$

Also $|x| = -x > x$, if $x < 0$

Thus in either case $|x|$ is greater of the two numbers, x and $-x$,

i.e., $|x| = \max(x, -x)$.

Corollary 1. $|-x| = \max(-x, -(-x))$

$= \max(-x, x) = |x|$

$\therefore$ $|-x| = |x|$

Corollary 2. $|x| = \max(x, -x) \geq x$

$\therefore$ $|x| \geq x$

Theorem 1.4. $-|x| = min\,(x, -x)$

Now $-|x| = -x < x$, if $x > 0$

Also $-|x| = -(-x) = x < -x$, if $x < 0$

Thus in either case $-|x|$ is smaller of the two numbers x and $-x$,

i.e., $-|x| = \min(x, -x)$.

Corollary. $-|x| = \min(x, -x) \leq x$.

$\therefore$ $-|x| \leq x$

Theorem 1.5. *If* $x, y \in \mathbf{R}$, *then*

(*i*) $|x|^2 = x^2 = |-x|^2$

(*ii*) $|xy| = |x| \cdot |y|$

(*iii*) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, *provided* $y \neq 0$.

(*i*) For $x \geq 0$,

$$|x| = x \Rightarrow |x|^2 = x^2$$

For $x < 0$.

$$|x| = -x \Rightarrow |x|^2 = (-x)^2 = x^2$$

Thus in either case $|x|^2 = x^2$

Similarly, $|-x|^2 = (-x)^2 = x^2$

Hence, $|x|^2 = x^2 = |-x|^2$

(*ii*) $$|xy|^2 = (xy)^2 = x^2y^2 = |x|^2 \cdot |y|^2 = (|x| \cdot |y|)^2$$

$\therefore$ $$|xy| = \pm |x| \cdot |y|$$

But since $|xy|$ and $|x| \cdot |y|$ are both non-negative, we take only the positive sign.

$\therefore$ $$|xy| = |x| \cdot |y|$$

(*iii*) $$\left|\frac{x}{y}\right|^2 = \left(\frac{x}{y}\right)^2 = \frac{x^2}{y^2} = \frac{|x|^2}{|y|^2} = \left(\frac{|x|}{|y|}\right)^2$$

But since $\left|\frac{x}{y}\right|$ and $\frac{|x|}{|y|}$ are both non-negative, therefore taking positive square root of both sides, we have

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}, \text{ when } y \neq 0.$$

Theorem 1.6. (*Triangle inequalities*). *For all real numbers x, y show that*

(*i*) $|x + y| \leq |x| + |y|$, *and*

(*ii*) $|x - y| \geq ||x| - |y||$.

(*i*) *First Method*:

$$|x + y|^2 = (x + y)^2 = x^2 + y^2 + 2xy$$

$$\leq |x|^2 + |y|^2 + 2|x| \cdot |y|$$

$$\left[\because \quad xy \leq |xy| = |x| \cdot |y|\right]$$

$$= (|x| + |y|)^2$$

Since $|x + y|$ and $|x| + |y|$ are both non-negative, therefore taking positive square roots of both sides, we have

$$|x + y| \leq |x| + |y|$$

Second Method: When $x + y \geq 0$.

$$|x+y| = x + y$$
$$\leq |x| + |y| \qquad [\because\ x \leq |x| \text{ and } y \leq |y|]$$

When $x + y < 0$,

$$|x + y| = -(x + y) = (-x) + (-y)$$
$$\leq |-x| + |-y| \qquad [\because\ -x \leq |-x| \text{ and } -y \leq |-y|]$$

But $|-x| = |x|, |-y| = |y|$

Thus in either case,

$$|x + y| \leq |x| + |y|.$$

(*ii*) *First Method*:

$$|x - y|^2 = (x - y)^2 = x^2 + y^2 - 2xy$$
$$\geq |x|^2 + |y^2| - 2|x| \cdot |y|$$
$$[\because\ -(xy) \geq -|xy| = -|x| \cdot |y|]$$
$$= (|x| - |y|)^2 = ||x| - |y||^2$$

Since, $|x - y|$ and $||x| - |y||$ are both non-negative, therefore taking the positive square root of both sides, we have

$$|x - y| \geq \| |x| - |y| \|.$$

Second Method:

Now $|x| = |(x - y) + y| \leq |x - y| + |y|$ [by part (*i*)]

$\therefore$ $|x - y| \geq |x| - |y|$ ...(1)

Again

$$|y| = |(y - x) + x| \leq |y - x| + |x|.$$

$\therefore$ $|y - x| \geq |y| - |x| = -(|x| - |y|)$

But $|y - x| = |x - y|$

$\therefore$ $|x - y| \geq -(|x| - |y|)$ ...(2)

From (1) and (2),

$$|x - y| \geq \max \{|x| - |y|, -(|x| - |y|)\}$$
$$= ||x| - |y||$$

Hence, $|x - y| \geq \| |x| - |y| \|$

Example 1.4. For real numbers $x, a, \varepsilon > 0$ show that

(i) $|x| < \varepsilon \Leftrightarrow -\varepsilon < x < \varepsilon,$

(ii) $|x - a| < \varepsilon \Leftrightarrow a - \varepsilon < x < a + \varepsilon.$

Solution. (i) $|x| = \max(x, -x) < \varepsilon$

$\Leftrightarrow x < \varepsilon \wedge -x < \varepsilon$

$\Leftrightarrow x < \varepsilon \wedge -\varepsilon < x$

$\Leftrightarrow -\varepsilon < x < \varepsilon$

(ii) $|x - a| = \max\{(x - a), -(x - a)\} < \varepsilon$

$\Leftrightarrow (x - a) < \varepsilon \wedge -(x - a) < \varepsilon$

$\Leftrightarrow x < a + \varepsilon \wedge a - \varepsilon < x$

$\Leftrightarrow a - \varepsilon < x < a + \varepsilon.$

Example 1.5. Show that for a bounded set S of real numbers there exists a number $G > 0$ such that

$$|x| \leq G, \quad \forall\ x \in S.$$

Solution. Let the set S, with or without zero, contain both positive and negative numbers. Since it is bounded, it is bounded both above and below. Let $k < 0$ be a lower bound and $K > 0$ be an upper bound. Now $\forall\ x \in S$,

$$x \leq K \Rightarrow |x| \leq |K| \qquad \text{when } x > 0$$

and

$$x \leq k \Rightarrow |x| \leq |k| \qquad \text{when } x < 0$$

So that $\forall\ \ x \in S, |x| \leq \max(|K|, |k|).$

Thus on taking a real number $G = \max(|K|, |k| + 1)$,

we have $$|x| < G, \qquad \forall\ \ x \in S.$$

EXERCISE

1. Show that

   (i) $|x| \geq -x.$

   (ii) $|x - y| = |y - x|.$

   (iii) $|x - y| \leq |x| + |y|.$

2. If $x_1, x_2, x_3, ..., x_n$ are real numbers, then show that

   (i) $|x_1 + x_2 + ... + x_n| \leq |x_1| + |x_2| + ... + |x_n|.$

   (ii) $|x_1 x_2 \ ... \ x_n| = |x_1| \cdot |x_2| \ ... \ |x_n|.$

CHAPTER 2

Limit Points: Open and Closed Sets

2.1 INTRODUCTION

In this chapter, we shall study the concept of *neighbourhood* of a point, *open and closed sets*, and *limit points* of a set of real numbers and the *Bolzano-Weierstrass theorem*, which is one of the most fundamental theorems of Real Analysis and lays down a sufficient condition for the existence of *limit points* of a set. *We shall be dealing only with real numbers and sets of real numbers unless otherwise stated.*

2.1.1 Neighbourhood of a Point

A set $N \subset \mathbf{R}$ is called the **neighbourhood** of a point a, if there exists an open interval I containing a and contained in N,

i.e., $$a \in I \subset N$$

It follows from the definition that an open interval is a neighbourhood of each of its points. Though open intervals containing the point are not the only neighbourhoods of the point but they prove quite adequate for a discussion like ours and are more expressive of the idea of neighbourhoods as understood in ordinary language. We shall, therefore, whenever convenient, take the open interval $]a - \delta, a + \delta[$ where $\delta > 0$ as a neighbourhood of the point a.

Deleted Neighbourhoods

The set $\{x : 0 < |x - a| < \delta\}$,

i.e., an open interval $]a - \delta, a + \delta[$ from which the number a itself has been *excluded* or *deleted* is called a *deleted neighbourhood* of a.

Note. For the sake of brevity, we shall write neighbourhood as '**nbd**'.

ILLUSTRATIONS

1. The set **R** of real numbers is the neighbourhood of each of its points.
2. The set **Q** of rationals is not the *nbd* of any of its points.
3. The open interval $]a, b[$ is *nbd* of each of its points.
4. The closed interval $[a, b]$ is the *nbd* of each point of $]a, b[$ but is not a *nbd* of the end points a and b.

5. The null set ϕ is a *nbd* of each of its points in the sense that there is no point in ϕ of which it is not a *nbd*.

Example 2.1. A non-empty finite set is not a *nbd* of any point.

Solution. A set can be a *nbd* of a point if it contains an open interval containing the point. Since an interval necessarily contains an infinite number of points, therefore, in order that a set be a *nbd* of a point it must necessarily contain an infinity of points. Thus a finite set cannot be a *nbd* of any point.

Example 2.2. Superset of a *nbd* of a point x is also a *nbd* of x. *i.e.*, if N is a *nbd* of a point x and $M \supset N$, then M is also a *nbd* of x.

Example 2.3. Union (finite or arbitrary) of *nbds* of a point x is again a *nbd* of x.

Example 2.4. If M and N are *nbds* of a point x, then show that $M \cap N$ is also a *nbd* of x.

Solution. Since M, N are *nbds* of x, $\exists$ open intervals enclosing the points x such that

$$x \in \left]x - \delta_1, x + \delta_1\right[\subset M \text{ and } x \in \left]x - \delta_2, x + \delta_2\right[\subset N$$

Let $\delta = \min\left(\delta_1, \delta_2\right)$. Then

$$\left]x - \delta, x + \delta\right[\subset \left]x - \delta_1, x + \delta_1\right[\subset M$$

and

$$\left]x - \delta, x + \delta\right[\subset \left]x - \delta_2, x + \delta_2\right[\subset N$$

$$\Rightarrow \quad \left]x - \delta, x + \delta\right[\subset M \cap N$$

$$\Rightarrow \quad M \cap N \text{ is a } nbd \text{ of } x.$$

2.1.2 Interior Points of a Set

A point x is an *interior point* of a set S if S is a *nbd* of x. In other words, x is an interior point of S if $\exists$ an open interval $]a, b[$ containing x and contained in S,

i.e., $$x \in]a, b[\subset S.$$

Thus a set is a neighbourhood of each of its interior points.

Interior of a Set. The set of all interior points of a set is called the **interior** of the set. The interior of a set S is generally denoted by S^i.

Exercise 1. Show that the interior of the set **N** or **I** or **Q** is the null set, but interior of **R** is **R.**

Exercise 2. Show that the interior of a set S is a subset of S, *i.e.*, $S^i \subset S$.

2.1.3 Open Set

A set S is said to be **open** if it is a *nbd* of each of its points, *i.e.*, for each $x \in S$, there exists an open interval I_x such that

$$x \in I_x \subset S.$$

Thus *every point of an open set is an interior point*, so that for an open set S, $S^i = S$.

Evidently, $$S \text{ is open } \Leftrightarrow S = S^i$$

Of course the set is **not open** if it is not a *nbd* of at least one of its points or that there is at least one point of the set which is not an interior point.

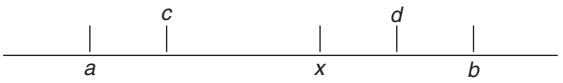
ILLUSTRATIONS

1. The set **R** of real numbers is an open set.
2. The set **Q** of rationals is not an open set.
3. The closed interval $[a, b]$, is not open for it is not a neighbourhood of the end points a and b.
4. The null set ϕ is open, for there is no point in ϕ of which it is not a neighbourhood.
5. A non-empty finite set is not open.
6. The set $\left\{\frac{1}{n}: n \in \mathbf{N}\right\}$ is not open.

Exercise. Give an example of an open set which is not an interval.

Example 2.5. Show that every open interval is an open set. Or, every open interval is a *nbd* of each of its points.

Solution. Let x be any point of the given open interval $]a, b[$ so that we have $a < x < b$.



Let c, d be two numbers such that

$$a < c < x, \text{ and } x < d < b$$

so that we have

$$a < c < x < d < b \Rightarrow x \in]c, d[\subset]a, b[.$$

Thus the given interval $]a, b[$ contains an open interval containing the point x, and is therefore a *nbd* of x.

Hence, the open interval is a *nbd* of each of its points and is therefore an open set.

Exercise. Show that every point of an open interval is its interior point.

Example 2.6. Show that every open set is a union of open intervals.

Solution. Let S be an open set and x_λ a point of S.

Since S is open, therefore $\exists$ an open interval I_{x_λ} for each of its points x_λ such that

$$x_\lambda \in I_{x_\lambda} \subset S \quad \forall\ x_\lambda \in S$$

Again the set S can be thought of as the union of singleton sets like $\{x_\lambda\}$,

i.e., $$S = \bigcup_{\lambda \in \Lambda} \{x_\lambda\}, \text{ where } \Lambda \text{ is the index set}$$

$\therefore$ $$S = \bigcup_{\lambda \in \Lambda} \{x_\lambda\} \subset \bigcup_{\lambda \in \Lambda} I_{x_\lambda} \subset S$$

$\Rightarrow$ $$S = \bigcup_{\lambda \in \Lambda} I_{x_\lambda}$$

Theorem 2.1. *The interior of a set is an open set.*

Let S be a given set, and S^i its interior.

If $S^i = \phi$, then S^i is open.

When $S^i \neq \phi$, let x be any point of S^i.

As x is an interior point of S, $\exists$ an open interval I_x such that

$$x \in I_x \subset S.$$

But I_x, being an open interval, is a *nbd* of each of its points.

$\Rightarrow$ every point of I_x is an interior point of I_x, and

$I_x \subset S \Rightarrow$ every point of I_x is interior point of S.

$\therefore$ $I_x \subset S^i$

$\Rightarrow$ $x \subset I_x \subset S^i \Rightarrow$ any point x of S^i is interior point of S^i

$\Rightarrow$ S^i is an open set.

Corollary. The interior of a set S is an open subset of S.

Theorem 2.2. *The interior of a set S is the largest open subset of S.*

or

The interior of a set S contains every open subset of S.

We know that the interior S^i of a set S is an open subset of S. Let us now proceed to show that any open subset S_1 of S is contained in S^i.

Let x be any point of S_1.

Since an open set is a *nbd* of each of its points, therefore S_1 is a *nbd* of x.

But S is a superset of S.

$\therefore$ S is also a *nbd* of x

$\Rightarrow$ x is an interior point of S

$\Rightarrow$ $x \in S^i$

Thus $x \in S_1 \Rightarrow x \in S^i$

$\therefore$ $S_1 \subset S^i$

Hence, every open subset of S is contained in its interior S^i.

$\Rightarrow$ S^i, the interior of S, is the largest open subset of S.

Corollary. Interior of a set S is the union of all open subsets of S.

Theorem 2.3. *The* **union** *of an arbitrary family of open sets is open.*

Let F be the union of an arbitrary family $\mathscr{F} = \{S_\lambda : \lambda \in \Lambda\}$ of open sets, Λ being an index set. To prove that F is open, we shall show that for any point $x \in F$, it contains an open interval containing x.

Let x be any point of F. Since F is the union of the members of $\mathscr{F}$, $\exists$ at least one member, say S_λ of $\mathscr{F}$ which contains x. Again, S_λ being an open set, $\exists$ an open interval I_x such that

$$x \in I_x \subset S_\lambda \subset F.$$

Thus the set F contains an open interval containing any point x of $F \Rightarrow F$ is an open set.

Theorem 2.4. *The* **intersection** *of any finite number of open sets is open.*

Let us consider two open sets *S, T.*

If $S \cap T = \phi$, it is an open set.

If $S \cap T \neq \phi$, let x be any point of $S \cap T$.

Now $x \in S \cap T \Rightarrow x \in S \wedge x \in T$.

$\Rightarrow$ S, T are *nbds* of x. $\quad [\because\ S, T \text{ are open}]$

$\Rightarrow$ $S \cap T$ is a *nbd* of x.

But since x is any point of $S \cap T$, therefore $S \cap T$ is a *nbd* of each of its points. Hence, $S \cap T$ is open.

The proof may of course be extended to a finite number of sets.

Note. The above theorem does not hold for the intersection of *arbitrary* family of open sets.

Consider for example the open sets

$$S_n = \left]-\frac{1}{n}, \frac{1}{n}\right[, n \in \mathbf{N}$$

Their intersection is the set $\{0\}$ consisting of the single point 0, and this set is not open.

2.2 LIMIT POINTS OF A SET

Definition 1. A real number ξ is a **limit point** of a set S $(\subset \mathbf{R})$ if every *nbd* of ξ contains an infinite number of members of *S.*

Thus ξ is a limit point of a set *S* if for any *nbd N* of ξ, $N \cap S$ is an infinite set.

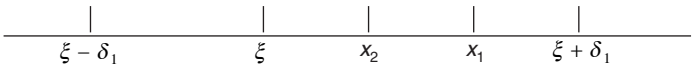
A limit point is also called a *cluster point*, *a condensation point* or an *accumulation point*.

A limit point of a set may or may not be a member of the set. Further it is clear from the definition that a finite set cannot have a limit point. Also it is not neccesary that an infinite set must possess a limit point. In fact a set may have no limit point, a unique limit point, a finite or an infinite number of limit points. A sufficient condition for the existence of a limit point is provided by *Bolzano-Weierstrass theorem* which is discussed in the next section. The following is another definition of a limit point.

Definition 2. A real number ξ is a **limit point** of a set S $(\subset \mathbf{R})$ if every *nbd* of ξ contains at least one member of *S* other than ξ.

The essential idea here is that the points of *S* different from ξ get 'arbitrarily close' to ξ or 'pile up' at ξ.

Evidently definition 1 implies definition 2. Let us now prove that definition 2 implies definition 1.



Let ξ be a limit point of the set S ($\subset \mathbf{R}$) such that every *nbd* of ξ contains at least one point of S other than ξ. Let $]\xi - \delta_1, \xi + \delta_1[$ be one such *nbd* of ξ which contains at least one point, say, $x_1 \neq \xi$ of S.

Let $|x_1 - \xi| = \delta_2 < \delta_1$. Now consider the *nbd* $]\xi - \delta_2, \xi + \delta_2[$ of ξ which by def. 2 of a limit point, must have one point, say, x_2 of S other than ξ.

By repeating the argument with the *nbd* $]\xi - \delta_3, \xi + \delta_3[$ of ξ where $\delta_3 = |x_2 - \xi|$ and so on, it follows that the *nbd* $]\xi - \delta_i, \xi + \delta_i[$ of ξ contains an infinity of members of *S.*

Hence, Def. 2 $\Rightarrow$ Def. 1.

It is instructive to note that a point ξ is not a limit point of a set *S* if $\exists$ even one *nbd* of ξ not containing any point of *S* other than ξ.

Exercise. Give a bounded set having (*i*) no limit point, (*ii*) infinite numbers of limit points.

Derived Sets. The set of all limit points of a set *S* is called the *derived set* of *S* and is denoted by S'.

ILLUSTRATIONS

1. The set **I** has no limit point, for a *nbd* $\left]m - \frac{1}{2}, m + \frac{1}{2}\right[$ of $m \in \mathbf{I}$, contains no point of **I** other than *m.* Thus the derived set of **I** is the null set ϕ.
2. Every point of **R** is a limit point, for, every *nbd* of any of its points contains an infinity of members of **R.** Therefore $\mathbf{R}' = \mathbf{R}$.
3. Every point of the set **Q** of rationals is a limit point, for, between any two rationals there exist an infinity of rationals. Further every irrational number is also a limit point of **Q** for between any two irrationals there are infinitely many rationals. Thus every real number is a limit point of **Q**, so that $\mathbf{Q}' = \mathbf{R}$.
4. The set $\left\{\frac{1}{n} : n \in \mathbf{N}\right\}$ has only one limit point, zero, which is not a member of the set.
5. Every point of the closed interval [*a*, *b*] is its limit point, and a point not belonging to the interval is not a limit point. Thus the derived set $[a, b]' = [a, b]$.
6. Every point of the open interval]*a*, *b*[is its limit point. The end points *a, b* which are not members of]*a*, *b*[are also its limit points. Thus

$$\left]a, b\right[' = [a, b].$$

Examples. Obtain the derived sets:

1. $\{x : 0 \leq x < 1\}$,
2. $\{x : 0 < x < 1, x \in \mathbf{Q}\}$,
3. $\left\{1, -1, 1\frac{1}{2}, -1\frac{1}{2}, -1\frac{1}{3}, \ldots\right\}$,
4. $\left\{1 + \frac{1}{n} : n \in \mathbf{N}\right\}$,

5. $\left\{\frac{1}{m}+\frac{1}{n}: m \in \mathbf{N}, n \in \mathbf{N}\right\}$.

2.2.1 A finite set has no limit point. Also we have seen that an infinite set may or may not have limit points. We shall now discuss a theorem which sets out sufficient conditions for a set to have limit points.

Bolzano-Weierstrass Theorem (for sets). *Every infinite bounded set has a limit point.*

Let S be any infinite bounded set and m, M its infimum and supremum respectively. Let P be a set of real numbers defined as follows:

$x \in P$ iff it exceeds at the most a finite number of members of S.

The set P is non-empty, for $M \in P$. Also M is an upper bound of P, for no number greater than or equal to M can belong to P. Thus the set P is non-empty and is bounded above. Therefore, by the order-completeness property, P has the supremum, say ξ. We shall now show that ξ is a limit point of S.

Consider any *nbd.* $]\xi-\varepsilon, \xi+\varepsilon[$ of ξ, where $\varepsilon > 0$.

Since ξ is the supremum of P, $\exists$ at least one member say η of P such that $\eta > \xi - \varepsilon$. Now η belongs to P, therefore it exceeds at the most a finite number of members of S, and consequently $\xi - \varepsilon\ (< \eta)$ can exceed at the most a finite number of members of S.

Again as ξ is the supremum of P, $\xi + \varepsilon$ cannot belong to P, and consequently $\xi + \varepsilon$ must exceed an infinite number of members of S.

Now $\xi - \varepsilon$ exceeds at the most a finite number of members of S and $\xi + \varepsilon$ exceeds infinitely many members of S.

$\Rightarrow\ ]\xi-\varepsilon, \xi+\varepsilon[$ contains an infinite number of members of S.

Consequently ξ is a limit point of S.

2.2.2 **Example 2.7.** If S and T are subsets of real numbers, then show that

(*i*) $S \subset T \Rightarrow S' \subset T'$, and

(*ii*) $(S \cup T)' = S' \cup T'$.

Solution. (*i*) If $S' = \phi$, then evidently $S' \subset T'$.

When $S' \neq \phi$, let $\xi \in S'$ and N be any *nbd* of ξ.

$\Rightarrow$ N contains an infinite number of members of S.

But $S \subset T$,

$\therefore$ N contains infinitely many members of T

$\Rightarrow$ ξ is limit point of T, *i.e.*, $\xi \in T'$.

Thus $\xi \in S' \Rightarrow \xi \in T'$.

Hence $S' \subset T'$.

(*ii*) Now $S \subset S \cup T \Rightarrow S' \subset (S \cup T)'$

and $T \subset S \cup T \Rightarrow T' \subset (S \cup T)'$

Consequently, $S' \cup T' \subset (S \cup T)'$ ...(1)

Now we proceed to show that $(S \cup T)' \subset S' \cup T'$.

If $(S \cup T)' = \phi$, then evidently $(S \cup T)' \subset S' \cup T'$.

When $(S \cup T)' \neq \phi$, let $\xi \in (S \cup T)'$.

Now ξ is a limit point of $(S \cup T)$, therefore, every *nbd* of ξ contains an infinite number of points of $(S \cup T) \Rightarrow$ every *nbd* of ξ contains infinitely many points of S or T or both.

$\Rightarrow$ ξ is a limit point of S or a limit point of T

$\Rightarrow$ $\xi \in S' \vee \xi \in T' \Rightarrow \xi \in S' \cup T'$.

Thus $\xi \in (S \cup T)' \Rightarrow \xi \in S' \cup T'$.

Consequently, $(S \cup T)' \subset S' \cup T'$ ...(2)

From (1) and (2) it follows that

$$(S \cup T)' = S' \cup T'$$

Thus the derived set of the union = the union of the derived sets.

(*ii*) **Aliter.** To show that $(S \cup T)' \subset S' \cup T'$

We may show that $\xi \notin S' \cup T' \Rightarrow \xi \notin (S \cup T)'$.

Now $\xi \notin S' \cup T'$ implies that ξ does not belong to either.

$\Rightarrow$ ξ is not a limit point of S or of T

$\therefore \exists$ *nbds*. N_1, N_2 of ξ such that N_1 contains no point of S other than ξ and N_2 contains no point of T other than possibly ξ.

Again, since $N_1 \cap N_2 \subset N_1$, $N_1 \cap N_2 \subset N_2$ therefore $\exists$ a *nbd*. $N_1 \cap N_2$ of ξ which contains no point other than ξ of S or of T and thus of $S \cup T$.

$\Rightarrow$ ξ is not a limit point $S \cup T$.

$\Rightarrow$ $\xi \notin (S \cup T)'$

Thus $\xi \notin S' \cup T' \Rightarrow \xi \notin (S \cup T)'$

so that $(S \cup T)' \subset S' \cup T'$

Example 2.8. (*i*) If S, T are subsets of **R**, then show that

$$(S \cap T)' \subset S' \cap T'.$$

(*ii*) Give an example to show that $(S \cap T)'$ and $S' \cap T'$ may not be equal.

Solution. (*i*) Now $S \cap T \subset S \Rightarrow (S \cap T)' \subset S'$

and $S \cap T \subset T \Rightarrow (S \cap T)' \subset T'$

Consequently, $(S \cap T)' \subset S' \cap T'$

(*ii*) Let $S =]1, 2[$ and $T =]2, 3[$, so that

$$S \cap T = \phi \Rightarrow (S \cap T)' = \phi' = \phi.$$

Also $S' = [1, 2], T' = [2, 3]$

$\therefore$ $S' \cap T' = \{2\}.$

Thus $(S \cap T)' \neq S' \cap T'.$

2.3 CLOSED SETS: CLOSURE OF A SET

2.3.1 A real number ξ is said to be an **adherent point** of a set S ($\subset \mathbf{R}$) if every *nbd* of ξ contains at least one point of *S*.

Evidently an adherent point may or may not belong to the set and it may or may not be a limit point of the set.

It follows from the definition that a number $\xi \in S$ is automatically an adherent point of the set, for, every *nbd* of a member of the set contains atleast one member of the set, namely the member itself. Further a number $\xi \notin S$ is an adherent point of S only if ξ is a limit point of *S*, for, every *nbd* of ξ then contains atleast one point of *S* which is other than ξ.

Thus the set of adherent points of *S* consists of *S* and the derived set S'.

The set of all adherent point of *S,* called the **closure** of *S*, is denoted by $\tilde{S}$, and is such that

$$\tilde{S} = S \cup S'.$$

ILLUSTRATIONS

1. $\tilde{\mathbf{I}} = \mathbf{I} \cup \mathbf{I}' = \mathbf{I} \cup \phi = \mathbf{I}.$
2. $\tilde{\mathbf{Q}} = \mathbf{Q} \cup \mathbf{Q}' = \mathbf{Q} \cup \mathbf{R} = \mathbf{R}.$
3. $\tilde{\mathbf{R}} = \mathbf{R} \cup \mathbf{R}' = \mathbf{R} \cup \mathbf{R} = \mathbf{R}$
4. $\tilde{\phi} = \phi \cup \phi' = \phi \cup \phi = \phi.$

2.3.2 Closed Sets

A set is said to be **closed** if each of its limit points is a member of the set.

In other words a set *S* is *closed* if no limit point of *S* exists which is not contained in *S*. In rough terms, a set is closed if its points do not get arbitrarily close to any point outside of it.

Thus a set *S* is *closed* iff

$$S' \subset S \quad \text{or} \quad \tilde{S} = S.$$

Consequently, a *closed set* is also defined as a set *S* for which

$$\tilde{S} = S.$$

It should be clearly understood that the concept of closed and open sets are neither mutually exclusive nor exhaustive. The word *not closed* should not be considered equivalent to *open*. Sets exist which are both open and closed, or which are neither open nor closed. The set consisting of points of]*a*, *b*] is neither open nor closed.

ILLUSTRATIONS

1. [*a*, *b*] is a set which is closed but not open.

2. The set $[0, 1] \cup [2, 3]$, which is not an interval, is closed.
3. The null set ϕ is closed for there exists no limit point of ϕ which is not contained in ϕ. As was shown earlier, ϕ is also open.
4. The set **R** of real numbers is open as well as closed.
5. The set **Q** is not closed, for $\mathbf{Q}' = \mathbf{R} \not\subset \mathbf{Q}$. Also it is not open.
6. $\left\{\frac{1}{n} : n \in \mathbf{N}\right\}$ is not closed, for it has one limit point, 0, which is not a member of the set. Also it is not open.
7. Every finite set A is a closed set, for its derived set $A' = \phi \subset A$.
8. A set A which has no limit point coincides with its closure, for $A' = \phi$ and $\tilde{A} = A \cup A' = A$.

2.3.3 Typical Examples

Example 2.9. Show that the set $S = \{x : 0 < x < 1, x \in \mathbf{R}\}$ is open but not closed.

Solution. The set S is the open interval $]0, 1[$.

$\therefore$ It contains a *nbd* of each of its points. Hence, it is an open set.

Again every point of S is a limit point. The end points 0 and 1 which are not members of the set are also limit points. Thus S is not closed.

Example 2.10. Show that the set

$$S = \left\{1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \ldots\right\}$$

is neither open nor closed.

Solution. The members of S heap or cluster near zero on both sides of it and every *nbd* of zero contains an infinite number of points of S. Thus $0 \notin S$ is a limit point $\Rightarrow$ S is not closed.

Again S is not open for it does not contain any *nbd* of any of its points. For example, a *nbd* $\left]\frac{1}{3} - \frac{1}{100}, \frac{1}{3} + \frac{1}{100}\right[$ of $\frac{1}{3}$ is not contained in the set. Hence, the set is not open.

Example 2.11. Show that the set

$$\left\{1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 1\frac{1}{3}, -1\frac{1}{3}, \ldots\right\}$$

is closed but not open.

Solution. 1 and –1 are the only limit points of the set and are in the set. Therefore, the set is closed.

Again all members of the set (except 1, –1) are not the interior points of the set. Thus the set is not open.

Hence, the set is closed but not open.

The relationship between closed and open sets is brought out by Theorem 2.5 that follows and is sometimes taken as the **definition** of a closed set.

2.3.4 Some Important Theorems

Theorem 2.5. *A set is closed iff its complement is open.*

Necessary. Let S be a closed set. We shall show that its complement $\mathbf{R} - S = T$ is open. Let x be *any* point of T.

$$x \in T \Rightarrow x \notin S.$$

Also, since S is closed, x cannot be a limit point of S. Thus $\exists$ a *nbd* N of x such that

$$N \cap S = \phi.$$

$\Rightarrow N \subset T \Rightarrow$ every point of T is an interior point.

Thus T is an open set.

Sufficient. Let S be a set whose complement $\mathbf{R} - S = T$ is open.

To show that S is closed, we shall show that every limit point of S is in S.

Let, if possible, a limit point ξ of S be not in S so that ξ is in T. As T is open, $\exists$ a *nbd* of ξ contained in T and thus containing no point of S.

$\therefore$ $\exists$ a *nbd* N of ξ which contains no point of S.

$\Rightarrow$ ξ is not a limit point of S; which is a contradiction.

Hence no limit point of S exists which is not in S.

$\therefore$ S is closed.

Theorem 2.6. *The intersection of an* **arbitrary** *family of closed sets is closed.*

Let F be the intersection set of an arbitrary family $\mathscr{F} = \{S_\lambda : \lambda \in \Lambda\}$ of closed sets, Λ being an index set.

If the derived set F' of F is ϕ, *i.e.*, when F is a finite set or an infinite set without limit points, then evidently it is closed.

When $F' \neq \phi$, let $\xi \in F'$, *i.e.* ξ be a limit point of F, so that every *nbd* of ξ contains infinitely many members of F and as such of each member S_λ of the family $\mathscr{F}$ of closed sets.

$\Rightarrow$ ξ is limit point of each closed set S_λ,

$\Rightarrow$ ξ belongs to each $S_\lambda \Rightarrow \xi \in \underset{\lambda \in \Lambda}{\cap} S_\lambda = F$.

Thus the set F is closed.

Note. We have given an independent proof but on taking complements, this theorem follows from theorem 2.3.

Theorem 2.7. *The union of* **two** *closed sets is a closed set.*

Let S, T be the two given closed sets and ξ a limit point of F, where $F = S \cup T$.

We have to show that $\xi \in F$, for then, the set F will be closed.

Let if possible $\xi \notin F$, thus $\xi \notin S \ \Lambda \ \xi \notin T$. Also as S, T are closed sets, the point ξ which does not belong to them, cannot be a limit point of either.

$\therefore$ $\exists$ *nbds* N_1 and N_2 of ξ such that

$$N_1 \cap S = \phi \wedge N_2 \cap T = \phi. \qquad ...(1)$$

Let $N_1 \cap N_2 = N$, where $\xi \in N$.

$\therefore$ From (1) it follows that

$$N \cap (S \cup T) = \phi \Rightarrow N \cap F = \phi.$$

Thus $\exists$ a *nbd N* of ξ which contains no point of *F*.

$\Rightarrow$ ξ is not a limit point of *F*, which is a contradiction.

Hence, no point not belonging to *F* can be its limit point, and consequently $F = S \cup T$ is a closed set.

Remarks 1. The theorem can be extended to the union of a finite number of sets. So we may restate the theorem as: *The union of a finite number of closed sets is closed.*

2. We have given an independent proof but the theorem follows from theorem 4 on taking complements.

3. The union of an arbitrary family of closed sets may not always be a closed set. For example, let

$$S_n = \left[a + \frac{1}{n}, a + 2 \right], \text{ for } n \in \mathbf{N} \wedge a \in \mathbf{R}.$$

Then $\underset{n \in \mathbf{N}}{\cup} S_n =]\, a, a + 2\,]$, which is not a closed set.

Theorem 2.8. *The derived set of a set is closed.*

Let S' be the derived set of a set *S*.

We have to show that the derived set S'' of S' is contained in S'.

Now if $S'' = \phi$, *i.e.*, when S' is either a finite set or an infinite set without limit points, then $S'' = \phi \subset S'$ and therefore S' is closed.

When $S'' \neq \phi$, let $\xi \in S''$, *i.e.*, ξ be a limit point of S'.

$\therefore$ Every *nbd N* of ξ contains at least one point $\eta \neq \xi$ of S'.

Again,

$\eta \in S' \Rightarrow \eta$ is a limit point of *S*

$\Rightarrow$ every *nbd* of η, *N* being such a *nbd*, contains infinitely many points of *S*.

Thus every *nbd N* (of ξ) contains an infinitely many points of *S*

$\Rightarrow$ ξ is a limit point of *S*, *i.e.*, $\xi \in S'$.

Consequently $\xi \in S'' \Rightarrow \xi \in S'$

$\therefore$ $S'' \subset S'$, *i.e.*, S' is a closed set.

Corollary 1. S'' is closed, and therefore the closure of S' is S'', *i.e.*, $\tilde{S} = S'$.

Corollary 2. For every set *S* the closure $\tilde{S}$ is closed.

We have simply to show that $(\tilde{S})' \subset \tilde{S}$.

Now, $\left(\tilde{S}\right)' = (S \cup S')' = S' \cup S'' = S' \subset \tilde{S}$. (ref. § 2.2)

Corollary 3. For every set S, $\tilde{\tilde{S}} = \tilde{S}$

$$\tilde{\tilde{S}} = \tilde{S} \cup (\tilde{S}') = \tilde{S}.$$

Theorem 2.9. *The supremum* (*infimum*) *of a bounded non-empty set* $S\ (\subset \mathbf{R})$, *when not a member of S, is a limit point of S.*

Let M be the supremum of the bounded set $S\ (\subset \mathbf{R})$, which must exist by the *order completeness* property of $\mathbf{R}$. If $M \notin S$, then for any number $\varepsilon > 0$, however small, $\exists$ at least one member x of S such that

$$M - \varepsilon < x < M.$$

Thus every *nbd* of M contains atleast one member x of the set S other than M. Hence M is a limit point of S.

Corollary. The supremum (infimum) M of a bounded set S is always a member of the closure $\tilde{S}$ of S.

When $M \in S$,

$$M \in S \Rightarrow M \in S \cup S' = \tilde{S}$$

When $M \notin S$,

$$M \notin S \Rightarrow M \in S' \Rightarrow M \in S \cup S' = \tilde{S}$$

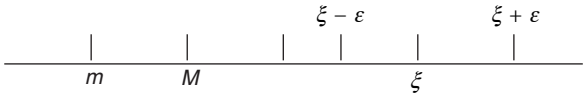
Consequently $M \in \tilde{S}$.

Theorem 2.10. *The derived set of a bounded set is bounded.*

Let m, M be the bounds of a set S.

It will now be shown that no limit point of S can be outside the interval $[m, M]$.

Let, if possible, $\xi > M$ be a limit point of S, and ε be a positive number such that $\varepsilon < \xi - M$.



Then since M is an upper bound of S, no member of S can lie in the interval $]\xi - \varepsilon, \xi + \varepsilon[$, therefore $\exists$ a *nbd* of ξ which contains no point of S so that ξ cannot be a limit point of S.

Hence S has no limit point greater than M.

Similarly it can be shown that no limit point of S is less than m.

Hence $$S' \subset [m, M].$$

Corollary. If S is bounded then so is its closure $\tilde{S}$.

$$S \subset [m, M] \Rightarrow S' \subset [m, M] \Rightarrow \tilde{S} = S \cup S' \subset [m, M].$$

Remark. If supremum M (infimum m) of S is not a member of S, then it is a limit point of S and in view of the above theorem, it is the greatest (least) member of S'.

However, if it is a member of S, then it is not necessarily a limit point of S, so that M (or m) may not be a member of $S' \subset [m, M]$. Thus M, m may not always be supremum and infimum of S' but they are always so for $\tilde{S} = S \cup S'$.

For example, for the set $S = \left\{-1, 1, -1\frac{1}{2}, 1\frac{1}{2}, -1\frac{1}{3}, 1\frac{1}{3}, \ldots\right\}$

$$m = -1\frac{1}{2},\ M = 1\frac{1}{2},\ S' = \{-1, 1\}$$

$$\inf S' = -1 \neq m,\ \text{Sup } S' = 1 \neq M$$

but

$$\inf \tilde{S} = m, \text{Sup } \tilde{S} = M .$$

Theorem 2.11. *The derived set* S' *of a bounded infinite set* S ($\subset \mathbf{R}$) *has the smallest and the greatest members.*

Since the set S is bounded, therefore S' is also bounded. Also S' is non-empty, for by Bolzano-Weierstrass theorem S has at least one limit point.

Now S' may be finite or inifinite.

When $S'(\neq \phi)$ is finite, evidently it has the greatest and the least members.

When S' is infinite, being bounded set of real numbers, by order-completeness property of **R**, it has the supremum G and the infimum g.

It will now be shown that G, g are limit points of S,

i.e., $$G \in S',\ \ g \in S'$$

Let us first consider G.

$$]G - \varepsilon, G + \varepsilon[,\ \varepsilon > 0,\ \text{be any } nbd \text{ of } G.$$

Now G being the supremum of S', $\exists$ at least one member ξ of S' such that $G - \varepsilon < \xi \leq G$.

Thus $$]G - \varepsilon, G + \varepsilon[\text{ is a } nbd \text{ of } \xi.$$

But ξ is a limit point of S, so that $]G - \varepsilon, G + \varepsilon[$ contains an infinity of members of S

$\Rightarrow$ any nbd $]G - \varepsilon, G + \varepsilon[$ of G contains an infinite number of members of S

$\Rightarrow$ G is a limit point of $S \Rightarrow G \in S'$

Similarly, it can be shown that $g \in S'$.

Thus $G \in S'$ and $g \in S'$, being supremum and infimum of S', are the greatest and the smallest members of S'.

The theorem may be restated as:

Every bounded infinite set has the smallest and the greatest limit points.

The smallest and greatest limit points of a set are called the **lower** and **upper limits of indetermination** or simply the **lower** and **upper limits** respectively of the set.

CHAPTER 3

Real Sequences

In this chapter we shall study a special class of functions whose domain is the set **N** of natural numbers and range a set of real numbers—the *Real Sequences*.

3.1 FUNCTIONS

Let A and B be two sets, and let there be a rule which associates to each member x of A, a member y of B.

Such a rule or a correspondence f under which to each element x of the set A, there corresponds exactly one element y of the set B, is called a *mapping* or a *function*.

Symbolically, we write

$$f: A \to B,$$

i.e., f is a mapping or a function of A *into* B.

The members of A are the elements to which some elements of B are to correspond. The set A is called the *Domain* of the function f. The set B contains all the elements which correspond to the elements of A, and is called to *co-domain* of f.

The unique element of B which corresponds to an element x of A is called the *image* of x or the value of the function at x and is denoted by $f(x)$; x is called the *preimage* of $f(x)$. It may be observed that while every element of the domain finds its image of B, there may be some elements in B which are not the image of any element of the domain A. The set of all those elements of the co-domain B which are the images of the elements of the domain A is called the *range* or the *range set* of the function f.

If f be a function with domain D and range R', then we say that f is a function from D onto R'.

In this book we shall be concerned only with the functions which are many to one, *i.e.* functions with respect to which one or more elements of the domain correspond to the same element of the range.

If members of the domain set are denoted by x and those of the range set by y, then $y = f(x)$; where $f(x)$ is the value of the function at x. However in the classical presentation of the subject instead of referring to f as a function we generally refer to $f(x)$ as function. In modern presentation of the subject such an erroneous terminology is being given up.

3.2 SEQUENCES

A function whose domain is the set **N** of natural numbers and range a set of real numbers **R** is called a *real sequence*. Thus a real sequence is denoted symbolically as $S: \mathbf{N} \to \mathbf{R}$.

Since we shall be dealing with real sequences only, we shall use the term *sequence* to denote a *Real sequence*.

NOTATION. Since the domain for a sequence is always **N**, a sequence is specified by the values S_n, $n \in \mathbf{N}$. Thus a sequence may be denoted as

$$\{S_n\},\ n \in \mathbf{N} \quad \text{or} \quad \{S_1, S_2, S_3, \ldots S_n, \ldots\}$$

The values S_1, S_2, S_3, ... are called the first, second, ... terms of the sequence. The mth and nth terms S_m and S_n for $m \neq n$ are treated as distinct terms even if $S_m = S_n$. Thus the terms of a sequence are arranged in a definite order as first, second, third, ... terms and the terms occurring at different positions are treated as distinct terms even if they have the same value. The number of terms in a sequence is always infinite.

In other words, we define a sequence as an ordered set of real numbers whose members can be put in a one-one correspondence with the set of natural numbers. However, a sequence may have only a finite number of distinct elements.

For example,

1. $\{S_n\} = \{(-1)^n\},\ n \in \mathbf{N}$.

   Here $S_1 = -1$, $S_2 = 1$, $S_3 = -1$, $S_4 = 1$, ... so that there are only two, 1, –1 distinct elements.

2. $\{S_n\} = \left\{\frac{1}{n}\right\},\ n \in \mathbf{N}$.

   Here $S_1 = 1$, $S_2 = \frac{1}{2}, S_3 = \frac{1}{3}, \ldots$

   All the elements are distinct.

ILLUSTRATIONS

1. $\{S_n\}$, where $S_n = \left(1 + \frac{1}{n}\right)^n$, $n \in \mathbf{N}$.
2. $\{S_n\}$, where $S_n = 1 + (-1)^n$, $n \in \mathbf{N}$.
3. $\{S_n\}$, where $S_n = 1,\ \forall\ n \in \mathbf{N}$.
4. $\left\{\frac{(-1)^{n-1}}{\lfloor \overline{n}!}\right\},\ n \in \mathbf{N}$.

3.2.1 The Range

The Range or the Range Set is the set consisting of all distinct elements of a sequence, without repetition and without regard to the position of a term. Thus the range may be a finite or an infinite set, without ever being the null set.

3.2.2 Bounds of a Sequence

Bounded above sequences

A sequence $\{S_n\}$ is said to be *bounded above* if there exists a real number K such that

$$S_n \leq K \quad \forall\ n \in \mathbf{N}$$

Bounded below sequences

A sequence $\{S_n\}$ is said to be *bounded below* if there exists a real number k such that

$$S_n \geq k \quad \forall\ n \in \mathbf{N}.$$

Bounded sequences

A sequence is said to be *bounded* when it is bounded both above and below. K and k are respectively the upper and the lower bounds of the sequence.

Evidently a sequence is bounded iff its range is bounded. Also the bounds of the range are the bounds of the sequence.

3.2.3 Convergence of Sequences

Definition 1. *A sequence* $\{S_n\}$ *is said to converge to a real number* l (*or to have the real number* l *as its limit*) *if for each* $\varepsilon > 0$, *there exists a positive integer* m (*depending on* ε) *such that* $|S_n - l| < \varepsilon$, *for all* $n \geq m$.

The fact is expressed by saying that the terms approach the value l or tend to l as n becomes larger and larger. The same thing expressed in symbols is

$$S_n \to l \text{ for } n \to \infty$$

or

$$\lim_{n \to \infty} S_n = l.$$

The definition ensures that

(*i*) From some stage onwards the difference between S_n and l can be made less than any preassigned positive number ε, however small, *i.e.*, given any positive real number ε, no matter however small, $\exists$ a positive integer m (finite) such that mth term onwards, S_n becomes and remains arbitrarily close to l, *i.e.* l is a limit point of the sequence.

(*ii*) For any $\varepsilon > 0$, at the most a finite number of terms (depending on the choice of ε) of the sequence can lie outside $]l - \varepsilon, l + \varepsilon[$, *i.e.* there is at the most a finite number of n's for which

$$S_n \leq l - \varepsilon \text{ and } S_n \geq l + \varepsilon.$$

(*iii*) Since $l - \varepsilon < S_n < l + \varepsilon$ for all $n \geq m$, therefore $S_n < l + \varepsilon$, for infinite number of terms, *i.e.*, infinite number of terms lie to the left of $l + \varepsilon$, or to the right of $l - \varepsilon$.

It may therefore be observed that if we can find even one $\varepsilon > 0$ for which infinitely many terms of the sequence lie outside $]l - \varepsilon, l + \varepsilon[$, then the sequence cannot converge to l.

3.2.4 Some Theorems

Theorem 3.1. *Every convergent sequence is bounded.*

Let a sequence $\{S_n\}$ converge to the limit l.

Let $\varepsilon > 0$ be a given number, so that $\exists$ a positive integer m such that

$$|S_n - l| < \varepsilon \quad \forall\ n \geq m$$

$$\Leftrightarrow l - \varepsilon < S_n < l + \varepsilon \quad \forall\ n \geq m.$$

Let

$$g = \min \{l - \varepsilon, S_1, S_2, \ldots, S_{m-1}\}.$$

$$G = \max \left\{ l + \varepsilon, S_1, S_2, ..., S_{m-1} \right\}.$$

Thus, we have

$$g \leq S_n \leq G \quad \forall\ n.$$

Hence, $\{S_n\}$ is a bounded sequence.

Theorem 3.2. *A sequence cannot converge to more than one limit.*

Let, if possible, a sequence $\{S_n\}$ converge to two real numbers l and $l'(l \neq l')$. Let us select $\varepsilon = \frac{1}{3}|l - l'| > 0.$

Let the sequence converges to l and l'; therefore, there exist positive integers m_1, m_2 such that

$$|S_n - l| < \varepsilon, \quad \forall\ n \geq m_1, \qquad ...(1)$$

and

$$|S_n - l'| < \varepsilon\ , \forall\ n \geq m_2. \qquad ...(2)$$

Now from (1) $S_n \in]\, l - \varepsilon, l + \varepsilon[, \forall\ n \geq m_1$, so that at the most a finite number of terms of the sequence can lie outside $]l - \varepsilon, l + \varepsilon[$.

Also $]l - \varepsilon, l + \varepsilon\,[\cap]\, l' - \varepsilon, l' + \varepsilon\,[= \phi$

$\therefore\ ]l' - \varepsilon, l + \varepsilon[$ can contain at most a finite number of terms of the sequence; which contradicts (2). Hence, the sequence cannot converge to two limits.

It may be seen from the definition that the number to which a sequence converges is a limit point of the sequence. Consequently, the unique limit to which the sequence converges is called *the limit point* or the **limit** of the sequence. Symbolically, we write

$$\lim_{n \to \infty} S_n = l \ \text{ or } \ S_n \to l \text{ as } n \to \infty,$$

or simply

$$\lim S_n = l.$$

Thus, *Theorems* 1 and 2 may be stated in a combined form as:

Theorem 3.3. *Every convergent sequence is bounded and has a unique limit.*

3.3 LIMIT POINTS OF A SEQUENCE

A real number ξ is said to be a **limit point** of a sequence $\{S_n\}$. If every neighbourhood of ξ contains an infinite number of members of the sequence.

Thus ξ is a limit point of a sequence if given any positive number ε, however small, $S_n \in]\xi - \varepsilon, \xi + \varepsilon[$ for an infinite number of values of n.

i.e., $|S_n - \xi| < \varepsilon,$ for infinitely many values of n.

Intuitively, it means that S_n is arbitrarily close to ξ for an infinite number of values of n or that infinitely many terms of the sequence occur very close to ξ.

As in a set, a *limit point* is also called a *cluster point* or a *point of condensation*.

Thus a number ξ is *not a limit point* of the sequence $\{S_n\}$ if $\exists$ a number $\varepsilon > 0$ such that $S_n \in]\xi - \varepsilon, \xi + \varepsilon[$ for at the most a finite number of values of n.

Note. A more intuitive but rigorous way of finding a limit point l is to see if the terms get 'closer and closer' to l. This will provide a 'guess' as to the limit point, after which the definition is applied to see if the guess is correct.

It is clear from the definition that a limit point of the range set of a sequence is also a limit point of the sequence. But the converse may not always be true. It may happen that the limit point ξ of a sequence is such that $S_n = \xi$ for an infinitely many values of n, so that automatically $S_n \in]\xi - \varepsilon, \xi + \varepsilon[$ for an infinity of values of n. In such a situation ξ is just one element in the range set and as such fails to be a limit point thereof. However, if $S_n \in]\xi - \varepsilon, \xi + \varepsilon[$ for an infinite number of values of n and $S_n = \xi$ for at the most a finite number of values of n, then a limit point ξ of the sequence is as well a limit point of the range.

ILLUSTRATIONS

1. The constant sequence $\{S_n\}$, where $S_n = 1, \ \forall\ n \in \mathbf{N}$, has the only limit point 1. The range is $\{1\}$ and has no limit point.
2. The sequence $\{S_n\}$, where $S_n = \frac{1}{n}, n \in \mathbf{N}$, has 0 as a limit point which is as well a limit point of the range $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}$.
3. 0 and 2 are the only limit points of the sequence $\{S_n\}$, where $S_n = 1 + (-1)^n, \ n \in \mathbf{N}$. The range set $\{0, 2\}$ has no limit point.
4. 1 and –1 are the two limit points of the sequence $\{S_n\}$, where $S_n = (-1)^n, \ n \in \mathbf{N}$. The range set $\{1, -1\}$ has no limit point.
5. 1 and –1 are the two limit points of the sequence $\{S_n\}$, where $S_n = (-1)^n \left(1 + \frac{1}{n}\right), n \in \mathbf{N}$, which are also the limit points of the range.

3.3.1 Existence of Limit Points

Since members of a sequence form a set (the range), all the theorems relating to bounds and limit points of sets hold also for sequences of members with suitable modifications. It is not necesssary, therefore, to list these again. However, we give a proof of one of the most fundamental theorems for sequences.

Bolzano-Weierstrass Theorem

Every bounded sequence has a limit point.

Let $\{S_n\}$ be a bounded sequence and $S = \{S_n : n \in \mathbf{N}\}$ be its range. Since the sequence is bounded, therefore its range set S is also bounded.

There are two possibilities:

(*i*) S is finite, (*ii*) S is infinite.

(*i*) If S is finite, then there must exist at least one member $\xi \in S$ such that $S_n = \xi$ for an infinite number of values of n. This means that every neighbourhood $]\xi - \varepsilon, \xi + \varepsilon[$ of ξ, contains $S_n (= \xi)$ for an infinite number of values of n.

Thus, ξ is a limit point of the sequence $\{S_n\}$.

(*ii*) When S is infinite, since it is bounded, it has by Bolzano-Weierstrass theorem (for sets), at least one limit point, say ζ.

Again, since ζ is a limit point of S, therefore every neighbourhood $]\xi - \varepsilon, \xi + \varepsilon[, \varepsilon > 0$ of ζ contains an infinity of members of S, *i.e.*, $S_n \in]\zeta - \varepsilon, \zeta + \varepsilon[$ for an infinity of values of n. Hence ζ is a limit point of the sequence.

Note. The converse of the theorem is not always true, for there do exist unbounded sequences having only one real limit point.

For example, {1, 2, 1, 4, 1, 6, …} has a unique limit point 1, but is not bounded above.

3.3.2 We have seen that a bounded sequence has at least one limit point. It may have one limit point, a finite or an infinite number of limit points. A number of questions arise. It may be asked, 'what is the greatest or the least limit point or whether the greatest or the least limit point exists at all?' In an attempt to answer such questions we now proceed to prove that such points do exist for bounded sequences.

Theorem 3.4. *The set of the limit points of a bounded sequence has the greatest and the least members.*

Let $\{S_n\}$ be a bounded sequence and S its range set. The set S is bounded and consequently its derived set S' is also bounded (Theorem 2.10, Ch. 2).

Let T be the set of limit points of the sequence. T is non-empty, for, by Bolzano-Weierstrass theorem the sequence has at least one limit point. Again, since T consists of the limit points of S (*i.e.*, the derived set S') and *those points of S* which are not the limit points of S but are limit points of the sequence. Therefore, T is bounded.

T may, however, be finite or infinite.

When $T(\neq \phi)$ is finite, it evidently has the greatest and the least members.

When T is infinite, being bounded set of real numbers, by the order-completeness property of real numbers, it has the supremum M and the infimum m, say.

It will now be shown that M and m are the limit points of the sequence *i.e.*, $M \in T, m \in T$.

Let us first consider M.

$$]M - \varepsilon, M + \varepsilon[, \varepsilon > 0, \text{ be any } nbd \text{ of } M.$$

Since M is the supremum of T, therefore $\exists$ at least one member ξ of T such that $M - \varepsilon < \xi \leq M$.

Thus $]M - \varepsilon, M + \varepsilon[$ is a *nbd* of ξ.

But ξ is a limit point of the sequence, so that $]M - \varepsilon, M + \varepsilon[$ contains an infinite of members of the sequence.

$\Rightarrow$ *nbd* $]M - \varepsilon, M + \varepsilon[$ of M contains an infinite number of members of the sequence

$\Rightarrow$ M is a limit point of the sequence

$\Rightarrow$ $M \in T$

Similarly, it may be shown that $m \in T$.

Thus $M \in T$, $m \in T$ and being the supremum and infimum of T are the greatest and the least members of T respectively.

Thus *a bounded sequence has the greatest and the least limit points* (one farthest to the right and other farthest to the left).

3.3.3* The greatest and the smallest of the limit points of a (bounded) sequence are respectively called the **upper limit** and the **lower limit** or the **limit superior** and the **limit inferior** of the sequence. They are generally denoted as

$$\lim_{n\to\infty} S_n, \quad \lim_{n\to\infty} S_n$$

or

$$\lim_{n\to\infty} \sup S_n, \quad \lim_{n\to\infty} \inf S_n, \text{ respectively.}$$

A bounded sequence $\{S_n\}$ for which the upper and the lower limits coincide with a real number l,

i.e.,

$$\lim_{n\to\infty} \sup S_n, = \lim_{n\to\infty} \inf S_n = l,$$

is said to converge to l, a fact symbolically expressed as

$$\lim_{n\to\infty} S_n = l.$$

Thus, we say that a bounded sequence with a unique limit point is convergent.

ILLUSTRATIONS

1. The sequence $\{S_n\}$, where $S_n = (-1)^n$, $n \in \mathbf{N}$, is bounded, for $-1 \le S_n \le 1$, $\forall\ n \in \mathbf{N}$.
   Also –1, 1 are the only limit points.
   $\therefore$ Upper limit = 1, lower limit = –1.
2. The sequence $\{S_n\}$, where $S_n = 1 + (-1)^n$, $n \in \mathbf{N}$, is bounded, for $0 \le S_n \le 2$, $\forall\ n \in \mathbf{N}$.
   0, 2 are the only limit points.
   $\therefore$ Upper limit = 2, lower limit = 0.
3. The sequence $\{S_n\}$, where $S_n = \dfrac{(-1)^{n-1}}{n!}, n \in \mathbf{N}$ *i.e.*, $\left\{1, \dfrac{-1}{2!}, \dfrac{-1}{3!}, \dfrac{-1}{4!}, \ldots\right\}$ is bounded, for $-\dfrac{1}{2} \le S_n \le 1, \forall\ n; 0$ being the only limit point, the upper and the lower limits coincide with 0, and so $\lim_{n\to\infty} S_n = 0$.
4. The sequence $\{S_n\}$, where $S_n = n^2$, is {1, 4, 9, 16, 25, ...}. The sequence is bounded below but not above. There is no real limit point.

3.4 CONVERGENT SEQUENCES

A sequence may have no limit point, a unique limit point or a finite or an infinite number of limit points. Our interest lies chiefly in a bounded sequence with a unique limit point. Evidently such a sequence can have at the most a finite number of terms outside the interval $]l - \varepsilon, l + \varepsilon[$, $\varepsilon > 0$, however, small ε may be. For otherwise, by the *Bolzano-Weierstrass Theorem* the infinite number of outside terms will have another limit point. Further, the condition automatically ensures the existence of an infinite number of terms of the sequence within the interval. Such sequences are called *convergent sequences*.

Let us proceed to show that the converse of theorem 3 also holds.

3.4.1 Theorem 3.5. *Every bounded sequence with a unique limit point is convergent.*

Let l be the only limit point of a bounded sequence $\{S_n\}$, which surely exists by *Bolzano-Weierstrass Theorem*. Thus, for $\varepsilon > 0$, $S_n \in]l - \varepsilon, l + \varepsilon[$ for an infinite number of values of $n \in \mathbf{N}$.

l being the only limit point of the sequence there can exist only a finite number of values say m_1, m_2, ..., m_r of n such that the corresponding terms of the sequence do not belong to $]l - \varepsilon, l + \varepsilon[$. For otherwise, the infinitely many outside terms will have a limit point other than l.

Let $(m - 1)$ be the greatest of such exceptional values of n. Thus, we have

$$S_n \in]l - \varepsilon, l + \varepsilon[, \quad \forall\ n \geq m,$$

$$\Rightarrow \qquad |S_n - l| < \varepsilon, \quad \forall\ n \geq m.$$

Thus the sequence $\{S_n\}$ converges to its unique limit point l.

Theorems 3.4 and 3.5 may be stated in the combined form as.

Theorem 3.6. *A necessary and sufficient condition for the convergence of a sequence is that it is bounded and has a unique limit point.*

3.4.2 In view of the above discussion, we give below another equivalent definition of a convergent sequence.

Definition 2. A sequence is said to be convergent if it is bounded and has a unique limit point.

Thus for a convergent sequence, the 'limit superior' and the 'limit inferior' coincide and the common value is called the *limit* of the sequence.

Starting with the definition 2 for the convergence of a sequence the reader is advised to prove the following theorem.

Theorem 3.7. *A necessary and sufficient condition for a sequence* $\{S_n\}$ *to converge to l is that to each* $\varepsilon > 0$ *there corresponds a positive integer m such that*

$$|S_n - l| < \varepsilon, \quad \forall\ n \geq m.$$

3.5* NON-CONVERGENT SEQUENCES (DEFINITIONS)

A sequence which is not convergent is called **Divergent**.

(*a*) *Bounded Sequences*

A bounded sequence which does not converge, and has at least two limit points, is said to **oscillate finitely.**

(*b*) *Unbounded Sequences*

(*i*) If a sequence $\{S_n\}$ is unbounded on the left (below), then we say that $-\infty$ is a limit point of the sequence, and to each positive number G, however large, there corresponds a positive integer m, such that

$$S_n < -G, \quad \forall\ n \geq m,$$

i.e., the sequence has an infinity of terms below $-G$.

Also, then, $-\infty$ is the least limit point, so that

$$\lim_{n\to\infty} \inf\ S_n = -\infty.$$

(*ii*) If a sequence $\{S_n\}$ is unbounded on the right (above), then we say that $+\infty$ is a limit point of the sequence, and to each positive number G, however large, there corresponds a positive integer, m, such that

$$S_n > G, \quad \forall\ n \geq m,$$

i.e., the sequence has an infinity of terms above G. Also, then, $+\infty$ is the greatest limit point, so that $\lim_{n\to\infty} \sup\ S_n = +\infty$.

(*iii*) If a sequence $\{S_n\}$ is bounded above (on the right) but not below and besides $-\infty$, has *no other* limit point, then $-\infty$ is not only its least but also its greatest limit point, and so we equate the upper limit also to $-\infty$, *i.e.*,

$$\lim_{n\to\infty} \inf\ S_n = \lim_{n\to\infty} \sup\ S_n = \lim_{n\to\infty} S_n = -\infty.$$

The sequence is, then, said to **diverge to** $-\infty$.

(*iv*) If, finally, the sequence is bounded on the left (below) but not on the right (above) and besides $+\infty$, has *no other* limit point, then $+\infty$ is not only its greatest but also its least limit point. So we have

$$\lim_{n\to\infty} \inf\ S_n = \lim_{n\to\infty} \sup\ S_n = \lim_{n\to\infty} S_n = +\infty.$$

The sequence is, then, said to **diverge to** $+\infty$.

(*v*) An unbounded sequence is said to oscillate infinitely if it diverges neither to $+\infty$ nor to $-\infty$. Thus a bounded sequence either *converges* or else *oscillates finitely*, but an unbounded sequence either *diverges* to $+\infty$ or $-\infty$ or *oscillates infinitely*.

Note. ∞ is not considered here as a real number because we are not dealing with the extended real number system. In the later case the definitions need lot of modifications.

ILLUSTRATIONS

1. $\{1 + (-1)^n\}$ oscillates finitely.
2. $\left\{(-1)^n\left(1 + \frac{1}{n}\right)\right\}$ oscillates finitely.
3. $\{n^2\}$ diverges to $+\infty$.
4. $\{-2^n\}$ diverges to $-\infty$.
5. $\{n(-1)^n\}$ oscillates infinitely.
6. $\left\{\frac{(-1)^{n-1}}{n!}\right\}$ converges to the limit 0.

7. $\left\{1+\frac{1}{n}\right\}$ converges to the limit 1.

8. $\left\{1,2,\frac{1}{2},3,\frac{1}{3},...\right\}$ is bounded below but unbounded above, and has a limit point 0 besides $+\infty$,

$$\lim_{n\to\infty} \inf S_n = 0,\ \lim_{n\to\infty} \sup S_n = +\infty.$$

The sequence oscillates infinitely.

9. {1, 2, 3, 2, 5, 2, 7, 2, 3, 2, 11, 2, 13, ...},

where, $S_n = \begin{cases} 2, \text{ when } n \text{ is even,} \\ \text{lowest prime factor } (\neq 1) \text{ of } n, \text{ when } n \text{ is odd,} \end{cases}$

is bounded on the left but not on the right. It has infinite number of limit points 2, 3, 5, 7, 11, ..., so that

$$\lim_{n\to\infty} \inf\ S_n = 2,\ \lim_{n\to\infty} \sup S_n = +\infty.$$

The sequence oscillates infinitely.

10. The sequence $\left\{m+\frac{1}{n}\right\}$ where m, n are natural numbers, also oscillates infinitely, 1, 2, 3, ... being its limit points.

Solved Examples

Example 3.1. Show that $\lim_{n\to\infty} \frac{3+2\sqrt{n}}{\sqrt{n}} = 2.$

Solution. Let ε be any positive number.

$$\therefore \qquad \left|\frac{3+2\sqrt{n}}{\sqrt{n}} - 2\right| < \varepsilon, \text{ if } \left|\frac{3}{\sqrt{n}}\right| < \varepsilon \text{ or } n > \frac{9}{\varepsilon^2}.$$

Let m be a positive integer greater than $9/\varepsilon^2$.

Thus to $\varepsilon > 0, \exists$ a positive m, such that

$$\left|\frac{3+2\sqrt{n}}{\sqrt{n}} - 2\right| < \varepsilon, \quad \forall\ n \geq m.$$

$$\therefore \quad \lim_{n\to\infty} \frac{3+2\sqrt{n}}{\sqrt{n}} = 2.$$

Example 3.2. Show that $\lim_{n\to\infty} \sqrt[n]{n} = 1.$

Solution. Let $\sqrt[n]{n} = 1 + h_n$, where $h_n \geq 0$

$$\therefore \qquad n = (1+h_n)^n = 1 + nh_n + \tfrac{1}{2}n(n-1)h_n^2 + ... + h_n^n$$

$$> \tfrac{1}{2}n(n-1)h_n^2, \quad \forall\ n\ (\because\ h_n \geq 0)$$

$$\therefore \qquad h_n^2 < \frac{2}{n-1}, \text{ for } n \geq 2$$

or $$|h_n| < \sqrt{\frac{2}{n-1}}, \text{ for } n \geq 2.$$

Let ε be any positive number, then

$$|h_n| < \sqrt{\frac{2}{n-1}} < \varepsilon, \text{ when } n > 1 + 2/\varepsilon^2.$$

Let m be any positive integer greater than $1+2/\varepsilon^2$.

Thus for $\varepsilon > 0, \exists$ a positive integer m such that

$$\left|\sqrt[n]{n} - 1\right| = |h_n| < \varepsilon, \quad \forall\ n \geq m.$$

Hence $$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

Example 3.3. Show that the sequence $\{r^n\}$ converges iff $-1 < r \leq 1$.

Solution. (*i*) When $r > 1$, let $r = 1 + h$, $h > 0$.

$$\therefore \qquad r^n = (1+h)^n$$

$$> 1 + nh, \quad \forall\ n \in \mathbf{N}.$$

If $G > 0$ be any number however large, we have

$$1 + nh > G, \text{ if } n > \frac{G-1}{h}.$$

Let m be a positive integer greater than $\dfrac{G-1}{h}$,

$\therefore$ For $G > 0, \exists$ a positive integer m such that

$$r^n > G, \quad \forall\ n \geq m.$$

Hence the sequence diverges to ∞.

(*ii*) When $r = 1$, evidently $\lim r^n = 1$.

$\therefore$ The sequence converges to 1.

(*iii*) When $|r| < 1$, let $|r| = \dfrac{1}{1+h}$, where $h > 0$.

$$\therefore \qquad |r^n| = |r|^n = \frac{1}{(1+h)^n} \leq \frac{1}{1+nh}, \quad \forall\ n \in \mathbf{N}.$$

Let ε be any positive number, then

$$\frac{1}{1+nh} < \varepsilon, \text{ when } n > \left(\frac{1}{\varepsilon} - 1\right) \Big/ h$$

Let m be a positive integer greater than $\left(\frac{1}{\varepsilon} - 1\right) \Big/ h$,

$\therefore$ For $\varepsilon > 0, \exists$ a positive integer m, such that

$$\left| r^n \right| < \varepsilon, \quad \forall\ n \geq m.$$

Hence, $\{r^n\}$ converges to 0, *i.e.*,

$$\lim\ r^n = 0, \text{ when } |\, r\,| < 1.$$

(*iv*) When $r = -1$, the sequence $\{(-1)^n\}$ is bounded and has two limit points.

$\therefore$ the sequence oscillates finitely.

(*v*) When $r < -1$, let $r = -t$ so that $t > 1$.

Thus we get the sequence $\{(-1)^n t^n\}$, which has both positive and negative terms. The sequence is unbounded and the numerical values of the terms can be made greater than any number however large. Thus, it oscillates infinitely.

Hence, the sequence $\{r^n\}$ converges only when $-1 < r \leq 1$.

Note. The sequence $\{r^n\}$ converges to zero *iff* $|\, r\,| < 1$.

EXERCISE

Show that

1. $\lim \dfrac{2n-3}{n+1} = 2.$
2. The sequence $\{(-1)^n\}$ oscillates finitely.
3. The sequence $\{S_n\}$, where $S_n = 1 + \dfrac{(-1)^n}{n}$, converges.
4. $lim\ \dfrac{1+2+3+\ldots+n}{n^2} = \dfrac{1}{2}.$
5. $\lim \dfrac{1+3+5+\ldots+(2n-1)}{n^2} = 1.$
6. The sequence $\{n + (-1)^n n\}$ oscillates infinitely.
7. Theorem 3.7 can be used to test the convergence of a sequence to a given limit l, but in cases where limit l is not known, nor can any guess be made of the same, the following theorem, which involves only the terms of the sequence, is useful for determining whether a sequence converges or not.

3.6 CAUCHY'S GENERAL PRINCIPLE OF CONVERGENCE

Theorem 3.7 can be used to test the convergence of a sequence to a given limit l, but in cases where limit l is not known, nor can any guess be made of the same, the following theorem, which involves only the terms of the sequence, is useful for determining whether a sequence converges or not.

Theorem 3.8. *A necessary and sufficient condition for the convergence of a sequence* $\{S_n\}$ *is that, for each* $\varepsilon > 0$ *there exists a positive integer m such that*

$$\left| S_{n+p} - S_n \right| < \varepsilon, \ \forall\ n \geq m \wedge p \geq 1.$$

Necessary: Let the sequence be convergent and let l be its limit, so that for a given $\varepsilon > 0$, $\exists$ a positive integer m, such that

$$\left| S_n - l \right| < \frac{1}{2}\varepsilon, \ \forall\ n \geq m.$$

If $p \geq 1$, then $n + p > n \geq m$, and so

$$\left| S_{n+p} - l \right| < \frac{1}{2}\varepsilon, \ \forall\ n \geq m \wedge p \geq 1$$

$$\Rightarrow \quad \left| S_{n+p} - S_n \right| = \left| S_{n+p} - l + l - S_n \right|$$

$$\leq \left| S_{n+p} - l \right| + \left| l - S_n \right|$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon, \ \forall\ n \geq m \wedge p \geq 1.$$

Sufficient: To establish the convergence of the sequence as a consequence of the given conditions, we show first that the sequence is bounded and then, that it converges to a limit.

Now by the given condition for $\varepsilon = 1$, $\exists$ a positive integer m_0 such that

$$\left| S_{n+p} - S_n \right| < 1, \ \forall\ n \geq m_0 \wedge p \geq 1.$$

In particular, for $n = m_0$

$$\left| S_{m_0+p} - S_{m_0} \right| < 1, \ \forall\ p \geq 1.$$

i.e.,

$$S_{m_0} - 1 < S_{m_0+p} < S_{m_0} + 1, \ \forall\ p \geq 1.$$

Let $g = \min \left\{ S_1, S_2, ..., S_{m_0-1}, S_{m_0} - 1 \right\}$

$G = \max \left\{ S_1, S_2, ..., S_{m_0-1}, S_{m_0} + 1 \right\}$.

Then, $g \leq S_n \leq G, \ \forall\ n$.

Hence, the sequence is bounded and therefore by *Bolzano-Weierstrass Theorem* for sequences, it has at least one limit point, say l. We shall now show that the sequence converges to l,

i.e., $\lim S_n = l$.

Now by the given condition, for $\varepsilon > 0$, $\exists$ a positive integer m such that

$$\left| S_{n+p} - S_n \right| < \frac{1}{3}\varepsilon, \ \forall\ n \geq m \wedge p \geq 1.$$

In particular, for $n = m$,

$$\left| S_{m+p} - S_m \right| < \frac{1}{3}\varepsilon, \ \forall\ p \geq 1. \qquad ...(1)$$

As l is a limit point, $\exists$ an integer $m_1 > m$ such that

$$\left| S_{m_1} - l \right| < \frac{1}{3}\varepsilon. \qquad ...(2)$$

Also, since $m_1 > m$, therefore from (1), we have

$$\left| S_{m_1} - S_m \right| < \frac{1}{3}\varepsilon \qquad \ldots(3)$$

$$\therefore \quad \left| S_{m+p} - l \right| = \left| S_{m+p} - S_m + S_m - S_{m_1} + S_{m_1} - l \right|$$

$$\leq \left| S_{m+p} - S_m \right| + \left| S_m - S_{m_1} \right| + \left| S_{m_1} - l \right|$$

$$< \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon, \ \forall\ p \geq 1$$

$$\Rightarrow \quad \left| S_n - l \right| < \varepsilon, \ \forall\ n \geq m.$$

Hence, the sequence $\{S_n\}$ converges to l.

3.6.1 Cauchy Sequences

A sequence $\{S_n\}$ is called a *Cauchy sequence* or a *fundamental sequence* if for each $\varepsilon > 0$, $\exists$ a positive integer m, such that

$$\left| S_{n+p} - S_n \right| < \varepsilon, \ \forall\ n \geq m \wedge p \geq 1$$

or

$$\left| S_{n_1} - S_{n_2} \right| < \varepsilon, \ \forall\ n_1, n_2 \geq m.$$

Thus in the field of real numbers, a sequence is convergent iff it is a Cauchy sequence.

Note. A sequence cannot converge if even one $\varepsilon > 0$ can be found such that for every positive integer m.

$$\left| S_{n+p} - S_n \right| \nless \varepsilon, \ \forall\ n \geq m \wedge p \geq 1.$$

EXERCISE

1. If $\{a_n\}, \{b_n\}$ are two Cauchy sequences, then the sequences $\{a_n \pm b_n\}, \{a_n b_n\}$ and $\{a_n/b_n\}$ (if $b_n \neq 0$ for all n) are also Cauchy sequences.
2. Show that every Cauchy sequence is bounded. Is the converse true?

Example 3.4. Show that the sequence $\{S_n\}$, where $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ cannot converge.

Solution. Let us take $\varepsilon = \frac{1}{2}$, and $n = m$ and $p = m$ in Cauchy's General Principle of Convergence, so that

$$\left| S_{2m} - S_m \right| < \frac{1}{2}.$$

But $S_{2m} - S_m = \frac{1}{m-1} + \frac{1}{m-2} + \ldots + \frac{1}{2m},$

$$> \frac{1}{2m} + \frac{1}{2m} + \ldots + \frac{1}{2m} = \frac{m}{2m} = \frac{1}{2},$$

i.e., $\left| S_{2m} - S_m \right| > \frac{1}{2}$, which contradicts Cauchy's general principle of convergence.

Hence, the sequence cannot converge.

3.7 ALGEBRA OF SEQUENCES

It may not always be easy to prove the convergence of a sequence by direct application of the definition of convergence or by Cauchy's General Principle of Convergence. But it may enable us to evaluate the limit of a sequence whose terms can be expressed as a sum, difference, product or quotient of the corresponding terms of two convergent sequences. The sequence $\{S_n\}$ whose nth term is $a_n + b_n$, $a_n - b_n$, $a_n b_n$ or $\frac{a_n}{b_n}$ $(b_n \neq 0)$ is called the sum, difference, product or quotient of the sequences $\{a_n\}$ and $\{b_n\}$. We shall now discuss the convergence of such sequences.

Theorem 3.9. *If $\{a_n\}$, $\{b_n\}$ be two sequences such that $\lim a_n = a$, $\lim b_n = b$, then*

(*i*) $\lim (a_n \pm b_n) = \lim a_n \pm \lim b_n = a \pm b$,

(*ii*) $\lim (a_n b_n) = (\lim a_n)(\lim b_n) = ab$,

(*iii*) $\lim(a_n / b_n) = (\lim a_n / \lim b_n)$, *if* $b \neq 0, b_n \neq 0, \forall n$.

(*i*) Let $\varepsilon > 0$ be given.

Since $\lim a_n = a$ and $\lim b_n = b$,

$\therefore$ $\exists$ positive integers m_1 and m_2, respectively, such that

$$|a_n - a| < \frac{1}{2}\varepsilon, \ \forall\ n \geq m_1, \text{ and}$$

$$|b_n - b| < \frac{1}{2}\varepsilon, \ \forall\ n \geq m_2.$$

Thus, for $m = \max(m_1, m_2)$, we have

$$|a_n - a| < \frac{1}{2}\varepsilon, \ |b_n - b|, < \frac{1}{2}\varepsilon, \ \forall\ n \geq m$$

$$\therefore \quad \left|(a_n \pm b_n) - (a \pm b)\right| = \left|(a_n - a) \pm (b_n - b)\right| \qquad \ldots(i)$$

$$\leq |a_n - a| \pm |b_n - b|$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon, \forall\ n \geq m.$$

Hence

$$\lim (a_n \pm b_n) = a \pm b - \lim a_n \pm \lim b_n.$$

(*ii*) The two sequences $\{a_n\}$, $\{b_n\}$ being convergent, are bounded so that $\exists$ positive real numbers k, K such that

$$|a_n| \leq k, |b_n| \leq K, \ \forall\ n.$$

Now

$$|a_n b_n - ab| = |a_n(b_n - b) + b(a_n - a)|$$

$$\leq |a_n| \cdot |b_n - b| + |b| \cdot |a_n - a|$$

$$\leq k\,|b_n - b| + |b| \cdot |a_n - a|. \qquad \ldots(ii)$$

Let $\varepsilon > 0$ be given.

Since $\lim a_n = a$, $\lim b_n = b$, therefore there exist positive integers m_1, m_2, respectively, such that

$$|a_n - a| < \frac{\frac{1}{2}\varepsilon}{|b|+1}, \quad \forall\ n \geq m_1,$$

$$|b_n - b| < \frac{\frac{1}{2}\varepsilon}{k}, \qquad \forall\ n \geq m_2,$$

Then, for $m = \max(m_1, m_2)$, we have from (1)

$$|a_n b_n - ab| < \frac{1}{2}\varepsilon + \frac{|b|.\frac{1}{2}\varepsilon}{|b|+1} < \varepsilon, \forall\, n \geq m.$$

Hence

$$\lim (a_n b_n) = ab = (\lim a_n)\ (\lim b_n).$$

(*iii*) *Lemma.* To show that if $\lim b_n = b \neq 0$, then $\exists$ a positive number λ and a positive integer m_3 such that

$$|b_n| > \lambda, \quad \forall\ n \geq m_3.$$

Let us take $\varepsilon = \frac{1}{2}|b|$, so that there exists a positive integer m_3 such that

$$|b_n - b| < \frac{1}{2}|b|, \quad \forall\ n \geq m_3,$$

Thus

$$|b| - |b_n| \leq |b_n - b| < \frac{1}{2}|b|$$

$$\Rightarrow \qquad |b_n| \geq \frac{1}{2}|b| = \lambda \text{ (say)}, \quad \forall\ n \geq m_3. \qquad \ldots(iii)$$

Let us apply the Lemma to prove the main theorem.

Now

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| = \left|\frac{ba_n - ab_n}{bb_n}\right| = \left|\frac{b(a_n - a) - a\,(b_n - b)}{bb_n}\right|$$

$$\leq \frac{|b|\,|a_n - a| + |a|\,|b_n - b|}{|b|\,|b_n|}$$

$$\leq \frac{2}{|b|}|a_n - a| + \frac{2|a|}{|b|^2}|b_n - b|, \quad \forall\ n \geq m_3 \text{ (using } (i)).$$

Let $\varepsilon > 0$ be given.

Since $\lim a_n = a$, $\lim b_n = b$, therefore, $\exists$ positive integers m_1, m_2 such that

$$|a_n - a| < \frac{1}{4}|b|\,\varepsilon, \quad \forall\ n \geq m_1,$$

and

$$|b_n - b| < \frac{1}{4}\frac{|b|^2\varepsilon}{|a|+1}, \quad \forall\ n \geq m_2.$$

Thus, for $m = \max(m_1, m_2, m_3)$, we have

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon, \quad \forall\ n \geq m.$$

Hence

$$\lim\left(\frac{a_n}{b_n}\right) = \frac{a}{b} = \frac{\lim a_n}{\lim b_n}.$$

Notes. It may be noted that while the sum, difference, product and quotient under certain conditions, of two convergent sequences is convergent, the converse may not be true, *i.e.*, if the sequence $\{a_n \pm b_n\}$, $\{a_n b_n\}$ or $\left\{\frac{a_n}{b_n}\right\}$ is convergent, the component sequences $\{a_n\}$ and $\{b_n\}$ may not be convergent, however, it is not difficult to see that both shall behave alike.

For example consider the sequences $\{a_n\}$ and $\{b_n\}$.

(1) When $a_n = n^2$ and $b_n = -n^2$.

The sequence $\{a_n + b_n\}$ converges to zero and the sequence $\left\{\frac{a_n}{b_n}\right\}$ converges to -1, but the two sequences $\{a_n\}$, $\{b_n\}$ are divergent.

(2) If $a_n = b_n = (-1)^n$.

The sequence $\{a_n - b_n\}$ converges to zero, $\left\{\frac{a_n}{b_n}\right\}$ converges to 1 and $\{a_n b_n\}$ converges to 1, while both the sequences $\{a_n\}$ and $\{b_n\}$ oscillate finitely.

(3) If $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$.

The sequence $\{a_n + b_n\}$ converges to zero, $\{a_n b_n\}$ converges to -1 and $\left\{\frac{a_n}{b_n}\right\}$ converges to -1, but $\{a_n\}$ and $\{b_n\}$ are not convergent.

Example 3.5. Show that $\lim\dfrac{(3n+1)(n-2)}{n(n+3)} = 3.$

Solution. We know that the sequence $\left\{\frac{1}{n}\right\}$ converges to zero, *i.e.*, $\lim\dfrac{1}{n} = 0.$

Now
$$\lim\frac{(3n+1)(n-2)}{n(n+3)} = \lim\frac{\left(3+\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\left(1+\frac{3}{n}\right)}$$

$$= \frac{\lim\left(3 + \frac{1}{n}\right)\lim\left(1 - \frac{2}{n}\right)}{\lim\left(1 + \frac{3}{n}\right)} = 3.$$

EXERCISE

1. Show that

   (*i*) $\lim \dfrac{3 + 2\sqrt{n+1}}{\sqrt{n+1}} = 2,$

   (*ii*) $\lim \dfrac{2n^2 - 5}{3n^2 + 7n} = \dfrac{2}{3},$

   (*iii*) $\lim \dfrac{1 + 2 + 3 + \ldots + n}{n^2} = \dfrac{1}{2},$

   (*iv*) $\lim \dfrac{1 + 3 + 3 + \ldots + (2n-1)}{n^2} = 1,$

   (*v*) $\lim [\sqrt{n+1} - \sqrt{n}] = 0.$

2. Show that

   $$\lim a_n = a \Rightarrow \lim |a_n| = |a|.$$

   Also by considering $a_n = (-1)^n$ or $(-1)^n\left(1 + \dfrac{1}{n}\right)$, show that the converse is not always true.

   Is the converse true if $a = 0$?

3. If $\{a_n\}$ converges and $\{b_n\}$ diverges, show that

   (*i*) $\lim \dfrac{a_n}{b_n} = 0.$

   (*ii*) $\{a_n + b_n\}$ is divergent.

4. Given that $\lim a_n = a$, $\lim b_n = b$, and $\{S_n\}$ and $\{T_n\}$ are two sequences, where

   $$S_n = \max(a_n, b_n),$$
   $$T_n = \min(a_n, b_n).$$

   Show that the sequences $\{S_n\}$ and $\{T_n\}$ are convergent and that

   $$\lim S_n = \max(a, b),$$
   $$\lim T_n = \min(a, b).$$

   [*Hint:* $\max(a_n, b_n) = \dfrac{1}{2}(a_n + b_n) + \dfrac{1}{2}|a_n - b_n|$

   $\min(a_n, b_n) = \dfrac{1}{2}(a_n + b_n) - \dfrac{1}{2}|a_n - b_n|.$]

5. Use Cauchy's General Principle of Convergence to show that the following sequences are convergent:

(*i*) $\left\{\dfrac{n}{n+1}\right\}$, (*ii*) $\left\{\dfrac{(-1)^n}{n}\right\}$.

3.8 SOME IMPORTANT THEOREMS

Theorem 3.10. *If* $\lim a_n = a$ *and* $a_n \geq 0$, *for all* n, *then* $a \geq 0$.

Let, if possible, $a < 0$.

Since $\lim a_n = a$, therefore for a given $\varepsilon > 0$, $\exists$ a positive integer m, such that

$$|a_n - a| < \varepsilon, \qquad \forall\ n \geq m.$$

$$\Rightarrow \qquad a - \varepsilon < a_n < a + \varepsilon, \qquad \forall\ n \geq m.$$

Let us select $\varepsilon = -\dfrac{1}{2}a$, so that corresponding to $\varepsilon = -\dfrac{1}{2}a > 0$, $\exists$ a positive integer m_1 such that

$$a + \frac{1}{2}a < a_n < a - \frac{1}{2}a, \quad \forall\ n \geq m_1.$$

$$\Rightarrow \qquad \frac{3}{2}a < a_n < \frac{a}{2}, \qquad \forall\ n \geq m_1.$$

or $$a_n < \frac{a}{2}, \qquad \forall\ n \geq m_1.$$

Since $a < 0$, therefore it follows that $a_n < 0$, for $n \geq m_1$. This contradicts the fact that $a_n > 0$, for all n. Therefore, the supposition is wrong.

Hence,

$$a \nleq 0 \ \Rightarrow\ a \geq 0.$$

Theorem 3.11. *If* $\{a_n\}$, $\{b_n\}$ *are two sequences such that*

(*i*) $a_n \leq b_n$, $\forall\ n$ *and*

(*ii*) $\lim a_n = a$, $\lim b_n = b$, *then* $a \leq b$.

Let, if possible, $a > b$.

Let $a - b = 3\varepsilon$, so that the neighbourhoods $]b - \varepsilon, b + \varepsilon[$, $]a - \varepsilon, a + \varepsilon[$ of b and a, respectively, are disjoint.

Since $\{a_n\}$, $\{b_n\}$ converge to a and b, respectively, therefore corresponding to $\varepsilon > 0$, $\exists$ positive integers m_1 and m_2, respectively, such that

$$a - \varepsilon < a_n < a + \varepsilon, \qquad \forall\ n \geq m_1,$$

$$b - \varepsilon < b_n < b + \varepsilon, \qquad \forall\ n \geq m_2.$$

Let

$$m = \max\ (m_1, m_2)$$

$$\therefore \qquad a_n \in\ ]a - \varepsilon, a + \varepsilon[, \qquad \forall\ n \geq m,$$

$$b_n \in\ ]b - \varepsilon, b + \varepsilon[, \qquad \forall\ n \geq m.$$

Consequently

$$b_n < a_n, \qquad \forall\ n \geq m.$$

Which contradicts the fact that

$$a_n \leq b_n, \qquad \forall\ n.$$

Hence, our supposition is wrong and therefore $a \leq b$.

Exercise. Deduce theorem 3.11 from theorem 3.10 by considering the sequence $\{c_n\}$, where $c_n = b_n - a_n$.

Theorem 3.12. ***(Sandwich theorem)*** *If* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ *are three sequences such that*

(*i*) $a_n \leq b_n \leq c_n$, $\forall\ n$, ...(1)

and (*ii*) $\lim a_n = \lim c_n = l$

then $\lim b_n = l$.

Let $\varepsilon > 0$ be given.

Now since $\{a_n\}$, $\{c_n\}$ both converge to l, therefore $\exists$ positive integers m_1, m_2, such that

$$|a_n - l| < \varepsilon, \qquad \forall\ n \geq m_1. \qquad ...(2)$$

and

$$|c_n - l| < \varepsilon, \qquad \forall\ n \geq m_2. \qquad ...(3)$$

Let $m = \max(m_1, m_2)$.

then, for $n \geq m$, we have from (1), (2) and (3)

$$l - \varepsilon < a_n \leq b_n \leq c_n < l + \varepsilon.$$

$$\Rightarrow \qquad l - \varepsilon < b_n < l + \varepsilon, \qquad \forall\ n \geq m,$$

$$\Rightarrow \qquad |b_n - l| < \varepsilon, \qquad \forall\ n \geq m.$$

Hence $\lim b_n = l$.

Example 3.6. Show that the sequence $\{b_n\}$, where

$$b_n = \left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + ... + \frac{1}{(2n)^2}\right], \text{ converges to zero.}$$

Solution. Evidently, $\dfrac{n}{(2n)^2} \leq b_n \leq \dfrac{n}{n^2}$, $\forall\ n$,

$$\Rightarrow \qquad \frac{1}{4n} \leq b_n \leq \frac{1}{n}, \quad \forall\ n.$$

Now the sequences $\{a_n\}$, $\{c_n\}$, where $a_n = \dfrac{1}{4n}$ and $c_n = \dfrac{1}{n}$ are such that

(*i*) $a_n \leq b_n \leq c_n$, and

(*ii*) $\lim a_n = \lim c_n = 0$.

$\therefore \quad \lim b_n = 0$.

Exercise 1. Show that the sequence $\{b_n\}$, where

$$b_n = \left\{ \frac{1}{\sqrt{(n^2+1)}} + \frac{1}{\sqrt{(n^2+2)}} + ... + \frac{1}{\sqrt{(n^2+n)}} \right\} \text{ converges to 1.}$$

Exercise 2. Show that $\lim \left\{ \frac{1}{\sqrt{(n+1)}} + \frac{1}{\sqrt{(n+2)}} + ... + \frac{1}{\sqrt{2n}} \right\} = \infty.$

Theorem 3.13. **(*Cauchy's first theorem on limits*)**

If $\lim\limits_{n\to\infty} a_n = l$, *then* $\lim\limits_{n\to\infty} \left(\frac{a_1 + a_2 + ... + a_n}{n} \right) = l.$

Let $b_n = a_n - l$.

Now, since $\lim a_n = l$, therefore

$$\lim b_n = 0.$$

Also

$$\frac{a_1 + a_2 + ... + a_n}{n} = l + \frac{b_1 + b_2 + ... + b_n}{n},$$

so that we have to show that

$$\lim \frac{b_1 + b_2 + ... + b_n}{n} = 0, \text{ when } \lim b_n = 0.$$

Now, since $\{b_n\}$ is convergent, therefore it is bounded and hence $\exists$ a number K such that

$$|b_n| < K, \qquad \forall\ n.$$

Let $\varepsilon > 0$ be given. Since $\{b_n\}$ converges to zero, therefore $\exists$ a positive m such that

$$|b_n| < \frac{1}{2}\varepsilon, \quad \text{for } n \geq m,$$

Also

$$\left| \frac{b_1 + b_2 + ... + b_n}{n} \right| = \left| \frac{b_1 + b_2 + ... + b_m}{n} + \frac{b_{m+1} + ... + b_n}{n} \right|$$

$$\leq \frac{|b_1| + |b_2| + ... + |b_m|}{n} + \frac{|b_{m+1}| + ... + |b_n|}{n}$$

$$\leq \frac{mK}{n} + \frac{\varepsilon}{2}\frac{(n-m)}{n}, \quad \forall\ n \geq m,$$

$$< \frac{mK}{n} + \frac{\varepsilon}{2}.$$

Let m_1 be positive integer greater than $\frac{2mK}{\varepsilon}$, so that

$$\frac{mK}{n} < \frac{\varepsilon}{2}, \text{ where } n \geq m_1.$$

Thus, for $n > \max(m, m_1)$, we have

$$\left| \frac{b_1 + b_2 + ... + b_n}{n} \right| < \varepsilon.$$

$$\Rightarrow \qquad \lim \frac{b_1 + b_2 + ... + b_n}{n} = 0.$$

Hence, $\lim \dfrac{a_1 + a_2 + ... + a_n}{n} = l.$

Note. The converse of the theorem is not true.
Let $a_n = (-1)^n$, so that

$$\frac{a_1 + a_2 + ... + a_n}{n} = 0, \text{ if } n \text{ is even,}$$

$$= -\frac{1}{n}, \text{ if } n \text{ is odd.}$$

$\Rightarrow \lim \dfrac{a_1 + a_2 + ... + a_n}{n} = 0.$ But $\{a_n\}$ is not convergent.

Example 3.7. Show that

$$\lim_{n \to \infty} \left[\frac{1}{\sqrt{n^2 + 1}} + \frac{1}{\sqrt{n^2 + 2}} + ... + \frac{1}{\sqrt{n^2 + n}} \right] = 1.$$

Solution. Let

$$a_n = \frac{n}{\sqrt{n^2 + n}},$$

$$\therefore \qquad \lim a_n = \lim \frac{1}{\sqrt{1 + \frac{1}{n}}} = 1.$$

Thus by *Cauchy's first theorem on limits*, we have

$$\lim \frac{a_1 + a_2 + ... + a_n}{n} = 1.$$

or

$$\lim \frac{1}{n} \left[\frac{n}{\sqrt{n^2 + 1}} + \frac{n}{\sqrt{n^2 + 2}} + ... + \frac{n}{\sqrt{n^2 + n}} \right] = 1.$$

or

$$\lim\left[\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+...+\frac{1}{\sqrt{n^2+n}}\right]=1.$$

Example 3.8. Show that

$$\lim_{n\to\infty}\frac{1}{n}\left[1+2^{1/2}+3^{1/3}+...+n^{1/n}\right]=1.$$

Solution. Consider $a_n = n^{1/n}$.

Now $\lim a_n = \lim n^{1/n} = 1$.

$\therefore$ By *Cauchy's first theorem on limits*,

$$\lim\frac{a_1+a_2+...+a_n}{n}=1.$$

$$\Rightarrow \quad \lim\frac{1}{n}\left[1+2^{1/2}+3^{1/3}+...+n^{1/n}\right]=1.$$

EXERCISE

1. Show that

(*i*) $\lim\left[\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n+1}}+...+\frac{1}{\sqrt{2n}}\right]=\infty,$

(*ii*) $\lim\frac{1}{n}\left[1+\frac{1}{2}+...+\frac{1}{n}\right]=0,$

(*iii*) $\lim\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}+...+\frac{1}{(2n)^2}\right]=0.$

Theorem 3.14. *If* $\{a_n\}$ *be a sequence such that* $\lim\frac{a_{n+1}}{a_n}=l$, *where* $|\,l\,|<1$, *then* $\lim a_n = 0$.

Since $|\,l\,|<1$, we can choose a positive number ε so small such that $|l|+\varepsilon<1$.

Now since $\lim\frac{a_{n+1}}{a_n}=l$, therefore $\exists$ a positive integer m, such that

$$\left|\frac{a_{n+1}}{a_n}-l\right|<\varepsilon, \quad \forall\ n\geq m,$$

$$\Rightarrow\left|\frac{a_{n+1}}{a_n}\right|-|l|\leq\left|\frac{a_{n+1}}{a_n}-l\right|<\varepsilon$$

$$\Rightarrow\left|\frac{a_{n+1}}{a_n}\right|<|l|+\varepsilon=k \text{ (say), where } k<1.$$

Putting $n = m, m + 1, ..., (n - 1)$ in turn and multiplying, we get

$$\left|\frac{a_n}{a_m}\right| < k^{n-m}$$

$$\Rightarrow \qquad |a_n| < \frac{|a_m|}{k^m} \cdot k^n$$

But as $k < 1,\ k^n \to 0,$

Hence

$$\lim a_n = 0.$$

Theorem 3.15. *If* $\{a_n\}$ *be a sequence, such that* $\lim \frac{a_{n+1}}{a_n} = l > 1,$ *then* $\lim a_n = \infty.$

Since $l > 1$, we can choose a positive number ε, such that $l - \varepsilon > 1.$

Now, since $\lim \frac{a_{n+1}}{a_n} = l,$ therefore $\exists$ a positive integer m, such that $\forall\ n \geq m,$

$$\left|\frac{a_{n+1}}{a_n} - l\right| < \varepsilon,$$

$$\Rightarrow \qquad l - \varepsilon < \frac{a_{n+1}}{a_n} < l + \varepsilon$$

$$\Rightarrow \qquad \frac{a_{n+1}}{a_n} > l - \varepsilon = k \text{ (say) where } k > 1$$

Putting $n = m, m + 1, ..., (n - 1)$ in turn and multiplying, we get

$$\left|\frac{a_n}{a_m}\right| \geq \frac{a_n}{a_m} > k^{n-m}$$

$$\Rightarrow \qquad |a_n| > \frac{|a_m|}{k^m} k^n$$

But as $k > 1,\ k^n \to \infty$

$$\therefore \qquad \lim a_n = \infty$$

Example 3.9. Show that for any real number x, $\lim \frac{x^n}{n!} = 0.$

Solution. Let $a_n = \frac{x^n}{n!}$

$$\therefore \qquad \frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n} = \frac{x}{n+1},$$

so that

$$\lim \frac{a_{n+1}}{a_n} = 0 < 1.$$

Hence by theorem 3.14, $\lim a_n = 0$.

Example 3.10. Show that

$$\lim \frac{m(m-1)(m-2)\ldots(m-n+1)}{n!}\, x^n = 0, |x| < 1$$

Solution. Let

$$a_n = \frac{m(m-1)(m-2)\ldots(m-n+1)}{n!}\, x^n.$$

$$\therefore \qquad \lim \frac{a_{n+1}}{a_n} = \lim \left(\frac{m-n}{n+1}\right) x = \lim \frac{\frac{m}{n} - 1}{1 + \frac{1}{n}}\, x = -x$$

But $|-x| = |x| < 1$.

$\therefore$ By theorem 3.14, $\lim a_n = 0$.

Note. See examples 4.21 and 4.22 of Chapter 4.

3.9 MONOTONIC SEQUENCES

A sequence $\{S_n\}$ is said to be *monotonic increasing*, if $S_{n+1} \geq S_n \ \forall\ n$, and *monotonic decreasing* if $S_{n+1} \leq S_n \ \forall\ n$. It is said to be monotonic if it is either monotonic increasing or monotonic decreasing.

A sequene (S_n) is *strictly increasing* if $\forall\ n, S_{n+1} > S_n$ and *strictly decreasing* if $S_{n+1} < S_n$.

The importance of monotonic sequences lies in the fact that they cannot oscillate. They either converge or diverge. Following theorem will make the point clear.

Theorem 3.16. *A necessary and sufficient condition for the convergence of a monotonic sequence is that it is bounded.*

The condition is necessary for we know that every convergent sequence is bounded.

The condition is sufficient. Let a bounded sequence $\{S_n\}$ be monotonic increasing. Let S denotes its range, which is evidently bounded. By the completeness property, S has the least upper bound (the supremum), say M.

We shall show that $\{S_n\}$ converges to M.

Let ε be any pre-assigned positive number.

Now since $M - \varepsilon$ is a number less than the supremum M, there exists at least one member say S_m such that $S_m > M - \varepsilon$.

As $\{S_n\}$ is a monotonic increasing sequence,

$\therefore \qquad S_n \geq S_m > M - \varepsilon, \qquad \forall\ n \geq m$

Again, since M is the supremum,

$$S_n \leq M < M + \varepsilon, \qquad \forall\ n$$

Thus

$$M - \varepsilon < S_n < M + \varepsilon, \qquad \forall\ n \geq m,$$

$\Rightarrow \qquad |S_n - M| < \varepsilon \qquad \forall\ n \geq m$

$\Rightarrow \qquad \{S_n\}$ converges and $\lim S_n = M$.

We may similarly consider the case of a bounded monotonic decreasing sequence.

Corollary 1. A monotonic increasing bounded above sequence converges to its least upper bound and a monotonic decreasing bounded below to the greatest lower bound.

Corollary 2. Every monotonic increasing sequence which is not bounded above, diverges to $+\infty$.

Let $\{S_n\}$ be a monotonic increasing sequence, not bounded above. Let G be any real number however large.

Since the sequence $\{S_n\}$ is unbounded above the monotonic increasing, $\exists$ a positive integer m such that

$$S_m > G \text{ and } S_n \geq S_m, \quad \forall\ n \geq m$$

$\Rightarrow \qquad S_n > G, \quad \forall\ n \geq m$

Hence $\lim S_n = +\infty$.

3.9.1 Subsequences

If $\{S_n\} = \{S_1, S_2, S_3, ...\}$ be a sequence, then any infinite succession of its terms, picked out in any way (but preserving the original order), is called a *subsequence* of $\{S_n\}$, or, in other words if $\{n_k\}$ be a strictly monotonic increasing sequence of natural numbers, *i.e.*, $n_1 < n_2 < n_3 < ...$, then $\{S_{n_k}\}$ is a *subsequence* of the sequence $\{S_n\}$.

ILLUSTRATIONS

1. $\{S_2, S_4, S_6, ..., S_{2n}, ...\}$ is a subsequence of $\{S_n\}$.
2. $\{S_1, S_4, S_9, ..., S_n^2, ...\}$ is a subsequence of $\{S_n\}$.
3. $\{S_7, S_8, S_9, ...\}$ is a subsequence of $\{S_n\}$, which is obtained by removing a finite number of terms from the beginning of $\{S_n\}$.

Without going into a formal proof we state:

1. *A sequence $\{S_n\}$ converges to s if and only if its every subsequence converges to s. Similarly $\lim S_n = \infty(-\infty)$ if and only if every subsequence of $\{S_n\}$ tends to $\infty(-\infty)$.*
2. *If ξ is a limit point of a sequence $\{a_n\}$, then there exists a subsequence of $\{a_n\}$ which converges to ξ.*

Exercise. The subsequence $\{x, x^4, x^9, x^{16}, ..., x^{n^2}, ...\}$ of $\{x^n\}$ converges to zero if $|x| < 1$, for the sequence $\{x^n\}$ converges to zero for $|x| < 1$.

Example 3.11. Show that the sequence $\{S_n\}$, where $S_n = \left(1 + \frac{1}{n}\right)^n$, is convergent and that $\lim \left(1 + \frac{1}{n}\right)^n$ lies between 2 and 3.

Solution. Expanding by Binomial theorem, since n is a positive integer, we have

$$S_n = \left(1 + \frac{1}{n}\right)^n = 1 + n\frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + ... + \frac{n(n-1)...3 \cdot 2 \cdot 1}{n!} \cdot \frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + ... + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)...\left(1 - \frac{n-1}{n}\right)$$

Since each term beyond the first two terms on the R.H.S. is an increasing function of n, it follows that $\{S_n\}$ is a monotonic increasing sequence. Again since each bracket on the R.H.S. is positive, therefore, we have

$$2 < S_n < 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{n!}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^{n-1}} < 3$$

$$\Rightarrow \qquad 2 < S_n < 3$$

Thus $\{S_n\}$ is a bounded and monotonic increasing sequence and so has a limit, which is generally denoted by e.

$$\therefore \quad \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n = e, \text{ where } 2 < e < 3.$$

Example 3.12. Show that the sequence $\{S_n\}$, where

$$S_n = \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{n+n} \text{ is convergent.}$$

Solution. Now

$$S_{n+1} - S_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= \frac{1}{2(n+1)(2n+1)} > 3, \quad \forall\ n$$

$\therefore$ the sequence $\{S_n\}$ is monotonic increasing.

Again

$$S_n = \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{n+n}$$

$$< \frac{1}{n} + \frac{1}{n} + ... + \frac{1}{n} = 1$$

i.e., $0 < S_n < 1$

$\therefore$ The sequence is bounded.

Hence the sequence, being bounded and monotonic increasing, is convergent.

Example 3.13. Show that the sequence, $\{S_n\}$, where

$$S_n = \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!}, \text{ is convergent.}$$

Solution. Now $S_{n+1} - S_n = \frac{1}{n+1!} > 0, \forall\ n$

$\therefore$ The sequence $\{S_n\}$ is monotonic increasing.

Again,

$$S_n = \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!}$$

$$< 1 + \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^{n-1}}$$

$$= \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 2 - \frac{1}{2^{n-1}} < 2$$

$\Rightarrow$ $0 < S_n < 2$

$\therefore$ The sequence is bounded.

Thus the sequence $\{S_n\}$, being bounded and monotonic increasing, is convergent.

Example 3.14. Show that the sequence $\{S_n\}$, defined by the recursion formula $S_{n+1} = \sqrt{3S_n}$, $S_1 = 1$, converges to 3.

Solution. The terms of the sequence $\{S_n\}$ are

$$1, \sqrt{3}, \sqrt{3\sqrt{3}}, \sqrt{3\sqrt{3\sqrt{3}}}, ...$$

Clearly

$$S_2 > S_1$$
$$S_3 > S_2$$

Also

$$S_{m+1} > S_m \Rightarrow \sqrt{3S_{m+1}} > \sqrt{3S_m} \Rightarrow S_{m+2} > S_{m+1}.$$

Thus by *Mathematical Induction*, the sequence $\{S_n\}$ is monotonic increasing.

Again,

$$S_1 < 3$$

$$S_2 < 3$$

$$S_3 = \sqrt{3S_2} < 3$$

and

$$S_m < 3 \Rightarrow \sqrt{3S_m} < \sqrt{3.3} \Rightarrow S_{m+1} < 3$$

So, again, by *Mathematical Induction*.

$$0 < S_n < 3, \ \forall\ n$$

Hence, the sequence $\{S_n\}$, being bounded and monotonic increasing, is convergent.
Let $\lim S_n = l$.

Since $\lim S_{n+1} = \lim \sqrt{3S_n}$

$\therefore$ $$l = \sqrt{3l} \Rightarrow l = 0 \text{ or } 3$$

But $l \neq 0$, since $S_n \geq 1, \ \forall\ n$
Hence, $\lim S_n = 3$.

Example 3.15. If $\{S_n\}$ be a sequence of positive real numbers such that $S_n = \frac{1}{2}(S_{n-1} + S_{n-2}), \forall\ n > 2,$ then show that $\{S_n\}$ converges. Also find $\lim S_n$.

Solution. If $S_1 = S_2$, then evidently

$$S_n = S_1, \ \forall\ n$$

so that the sequence converges to S_1.

When $S_1 \neq S_2$, let $S_1 < S_2$.

Putting $n = 3, 4, 5, ..., m$, in the relation $S_n = \frac{1}{2}(S_{n-1} + S_{n-2})$, we find that

$$\left.\begin{aligned} S_3 &= \frac{1}{2}(S_2 + S_1) \\ S_4 &= \frac{1}{2}(S_3 + S_2) \\ S_5 &= \frac{1}{2}(S_4 + S_3) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ S_m &= \frac{1}{2}(S_{m-1} + S_{m-2}) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \quad \ldots(1)$$

$\Rightarrow$

$$\begin{aligned} &S_1 < S_3 < S_2 \\ &S_3 < S_4 < S_2 \\ &S_3 < S_5 < S_4 \\ &S_5 < S_6 < S_4 \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

Thus it appears that

and
$$\left.\begin{aligned} S_1 < S_3 < S_5 < \ldots \\ S_2 > S_4 > S_6 > \ldots \end{aligned}\right\} \quad \ldots(2)$$

Now

$$S_{n+2} - S_n = \frac{1}{2}\left(S_{n+1} + S_n\right) - S_n$$

$$= \frac{1}{2}\left(S_{n+1} - S_n\right) \quad \ldots(3)$$

$$= \frac{1}{4}\left(S_n - S_{n-2}\right) \quad \ldots(4)$$

From (4) we easily see that

(*i*) The subsequence of odd terms is monotonic increasing,
i.e., $S_1 < S_3 < S_5 < \ldots$

(*ii*) The subsequence of even terms is monotonic decreasing,
i.e., $\ldots < S_6 < S_4 < S_2$.

Again from (3) when n is even, putting $n = 2m$, we get

$$S_{2m+2} - S_{2m} = \frac{1}{2}\left(S_{2m+1} - S_{2m}\right)$$

$\therefore$ $$S_{2m+2} < S_{2m} \Rightarrow S_{2m+1} < S_{2m}$$

but

$$S_{2m} < S_{2m-2} < \ldots < S_4 < S_2$$

$\therefore$ Every odd term is less than every even term, *i.e.*,

$$S_1 < S_3 < S_5 < \ldots < S_{2m+1} < S_{2m} < S_{2m-2} < \ldots < S_6 < S_4 < S_2$$

Thus the odd term subsequence $\{S_{2n+1}\}$ is monotonic increasing and is bounded above (by S_2) and is therefore convergent.

Similarly the even term subsequence $\{S_{2n}\}$ is convergent.

Let us now show that the two subsequences converge to the same number.

Let $\{S_{2n+1}\} \to l$ and $\{S_{2n}\} \to l'$.

From the recursion formula

$$S_{2m} = \frac{1}{2}\left(S_{2m-1} + S_{2m-2}\right)$$

on taking limits as $m \to \infty$, we get

$$l' = \frac{1}{2}\left(l + l'\right) \Rightarrow l = l'.$$

Thus both the subsequences of $\{S_n\}$ converge to l

$\Rightarrow$ the sequence $\{S_n\}$ converges to l.

Again from (1) by adding

$$S_k + \frac{1}{2}S_{k-1} = \frac{1}{2}\left(S_1 + 2S_2\right).$$

Taking limits when $k \to \infty$, we get

$$l + \frac{1}{2}l = \frac{1}{2}(S_1 + 2S_2)$$

$$\therefore \qquad l = \frac{1}{3}(S_1 + 2S_2)$$

Thus the sequence $\{S_n\}$ converges to $\frac{1}{3}(S_1 + 2S_2)$.

It may similarly be shown that the case $S_1 > S_2$ leads to the same result.

Example 3.16. If a sequence $\{S_n\}$ is defined by

$$S_n = \frac{S}{1 + S_{n-1}}, \text{ where } S > 0, S_1 > 0,$$

then show that the sequence converges to the positive root of the equation

$$x^2 + x - S = 0.$$

Solution. Now

$$S_n = \frac{S}{1 + S_{n-1}} \qquad \ldots(1)$$

$$S_n - S_{n-2} = \frac{S}{1 + S_{n-1}} - \frac{S}{1 + S_{n-3}}$$

$$= \frac{-S\left(S_{n-1} - S_{n-3}\right)}{\left(1 + S_{n-1}\right)\left(1 + S_{n-3}\right)} \qquad \ldots(2)$$

$$= \frac{S^2\left(S_{n-2} - S_{n-4}\right)}{\left(1 + S_{n-1}\right)\left(1 + S_{n-2}\right)\left(1 + S_{n-3}\right)\left(1 + S_{n-4}\right)} \qquad \ldots(3)$$

(3) shows that the even and odd terms form separate monotonic subsequences.

Again (2) shows that if odd terms form a monotonic decreasing subsequence, even terms will form a monotonic increasing subsequence and vice-versa.

Since every term of the sequence is positive,

$$\therefore \qquad S - S_n = \frac{SS_{n-1}}{1 + S_{n-1}} > 0$$

$$\Rightarrow \qquad 0 < S_n < S, \ \forall\ n$$

a result which could have been written directly from (1).

Thus the monotonic increasing subsequence is bounded above by S and the monotonic decreasing subsequence is bounded below by 0.

Hence, the two subsequences converge.

Let, if possible, the even term subsequence converge to l and the odd term subsequence to l'.

$\therefore$ Taking the limit, we get from (1)

(*i*) for n even, $l = \dfrac{S}{1+l'}$ or $ll' + l = S$

(*ii*) for n odd, $l' = \dfrac{S}{1+l}$ or $ll' + l' = S$

$\Rightarrow$ $l = l'$

Thus both the subsequences converge to the same number l,

$\Rightarrow$ the given sequence $\{S_n\}$ converges to l.

Again from (1), on proceeding to limits we get

$$l = \frac{S}{1+l} \Rightarrow l^2 + l - S = 0$$

$\therefore$ l is a root of $x^2 + x - S = 0$

where l is positive, for, every term of the sequence is positive.

Theorem 3.17 (*Nested-intervals*). *If a sequence of closed intervals* $[a_n, b_n]$ *is such that each member* $[a_{n+1}, b_{n+1}]$ *is contained in the preceding one* $[a_n, b_n]$ *and lim* $(b_n - a_n) = 0$, *then there is one and only one point common to all the intervals of the sequence*.

Since each interval member of the sequence is contained in the preceding one, therefore, we have

$$a_1 \le a_2 \le ... \le a_n \le ...$$

$$b_1 \ge b_2 \ge ... \ge b_n \ge ...,$$

so that sequence $\{a_n\}$ is monotonic increasing, and $\{b_n\}$ is monotonic decreasing. Also $a_n \le b_1$ and $b_n \ge a_1$, for all n, so that sequence $\{a_n\}$ is bounded above by b_1 and $\{b_n\}$ bounded below by a_1.

Thus both the sequences are convergent.

Let $\lim a_n = \xi$ and $\lim b_n = \eta$.

Again

$$0 = \lim (b_n - a_n) = \lim b_n - \lim a_n = \xi - \eta$$

$\Rightarrow$ $\xi = \eta$.

Obviously ξ is the upper bound of the sequence $\{a_n\}$ and lower bound of the sequence $\{b_n\}$, and hence

$$a_n \le \xi \le b_n, \ \forall\ n$$

so that ξ belongs to all the intervals.

Let, if possible, ξ_1, ξ_2 be two different points common to all the intervals, and let $\xi_1 < \xi_2$.

Then

$$a_n \le \xi_1 < \xi_2 \le b_n, \ \forall\ n$$

i.e.,

$$b_n - a_n \ge \xi_2 - \xi_1 \ne 0, \ \text{ for all } n,$$

which is a contradiction to the fact that $\xi_1 < \xi_2$ ($\because \lim (b_n - a_n) = 0$.)

Hence the result.

EXERCISE

1. Show that the sequence $\{S_n\}$, where
$$S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{(n-1)!}$$
is convergent.
2. Prove that the sequence $\{S_n\}$ defined by the recursion formula
$$S_{n+1} = \sqrt{7 + S_n},\ S_1 = \sqrt{7},$$
converges to the positive root of
$$x^2 - x - 7 = 0.$$
3. If $\{S_n\}$ be a sequence such that
$$S_{n+1} = 2 - \frac{1}{S_n},\ n \geq 1 \text{ and } S_1 = \frac{3}{2},$$
then show that the sequence $\{S_n\}$ is bounded and monotonic and converges to 1.
4. Given that $\{a_n\}$ is a sequence such that
$$a_2 \leq a_4 \leq a_6 \leq ... \leq a_5 \leq a_3 \leq a_1$$
and a sequence $\{b_n\}$, where $b_n = a_{2n-1} - a_{2n}$, converges to 0, then show that the sequence $\{a_n\}$ is convergent.
5. Let $\{a_n\}$ be a sequence, defined by
$$a_{n+1} = \frac{4 + 3a_n}{3 + 2a_n},\ n \geq 1 \quad a_1 = 1,$$
show that $\{a_n\}$ converges to $\sqrt{2}$.
6. If $\{a_n\}$ be a sequence of positive real numbers such that
$$a_n = \sqrt{a_{n-1}\, a_{n-2}},\ n > 2,$$
then show that the sequence converges to $(a_1 a_2^2)^{1/3}$.

CHAPTER 4

Infinite Series

4.1 INTRODUCTION

In this chapter we shall discuss the techniques of testing the *behaviour* of infinite series as regards convergence. The most important technique for series, all of whose terms are of the same sign (all positive or all negative), is to compare the given series with another suitably chosen series with known behaviour. So, first of all, comparison tests are discussed, and then some special tests for convergence are considered. *Leibnitz's test* for alternating series, *Abel's* and *Dirichlet's tests* for arbitrary term series, and *Dirichlet's* and *Riemann's* theorems on rearrangement of terms will be discussed in detail towards the end.

4.1.1 A *series* is the sum of the terms of a sequence. Thus if $u_1, u_2, u_3, \ldots$ is a sequence then the sum $u_1 + u_2 + u_3 + \ldots$ of all the terms is called an *infinite series* and is denoted by $\sum_{n=1}^{\infty} u_n$ or simply by $\sum u_n$.

Evidently we cannot just add up all the infinite number of terms of the series in the ordinary way and in fact it is not obvious that this kind of sum has any meaning. We thus start by associating with the given series, a sequence $\{S_n\}$, where S_n denotes the sum of the first n terms of the series. Thus

$$S_n = u_1 + u_2 + \ldots + u_n,$$

The sequence $\{S_n\}$ is called the *sequence of partial sums* of the series and the partial sums, $S_1 = u_1$, $S_2 = u_1 + u_2$, $S_3 = u_1 + u_2 + u_3$, and so on, may be regarded as approximations to the full infinite sum $\sum_{n=1}^{\infty} u_n$ of the series. If the sequence $\{S_n\}$ of partial sums converges, then the series is regarded as convergent, and $\lim S_n$ is said to be the *sum of the series*. If however, $\{S_n\}$ does not tend to a limit, we must take it that the sum of the infinite series does not exist. We express this fact by saying that the series does not converge. In fact *an infinite series is said to converge, diverge or oscillate according as its sequence of partial sums* $\{S_n\}$ *converges, diverges or oscillates*.

4.1.2 A Necessary Condition for Convergence

Theorem 4.1. *A necessary condition for convergence of an infinite series* $\sum u_n$ *is that* $\lim_{n\to\infty} u_n = 0$.

Let $S_n = u_1 + u_2 + \ldots + u_n$ so that $\{S_n\}$ is the sequence of partial sums.

Since the series converges, therefore, the sequence $\{S_n\}$ also converges.

Consequently

$$\lim_{n\to\infty} S_n = s(\text{say})$$

Now,

$$u_n = S_n - S_{n-1}, \quad n > 1$$

$$\therefore \quad \lim_{n\to\infty} u_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= s - s = 0$$

Hence for a convergent series

$$\lim u_n = 0$$

In other words, a series cannot converge if its *n*th term does not tend to zero,

Notes 1. It must be clearly understood that $\lim u_n = 0$ does not prove that a series in convergent, for there exist series which do not converge even though $\lim u_n = 0$. See Example 4.2 below.

2. However, $\lim u_n \neq 0$ proves that the series does not converge, see Example 4.1 below:

Example 4.1. Show that the series

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots \text{ is not convergent.}$$

Solution. Here $$u_n = \frac{n}{n+1}$$

$$\therefore \quad \lim u_n = \lim \frac{n}{n+1} = 1 \neq 0$$

Since $\lim u_n \neq 0$, therefore, the series is not convergent.

4.1.3 Cauchy's General Principle of Convergence for Series

A necessary and sufficient condition for the convergence of an infinite series $\sum_{n=1}^{\infty} u_n$ is that the sequence of its partial sums $\{S_n\}$ is convergent.

Therefore, a test for convergence of infinite series may be derived from our knowledge of sequences.

Theorem 4.2. *A series* $\sum u_n$ *converges iff to each* $\varepsilon > 0$, *there exists a positive integer m such that*

$$\left| u_{n+1} + u_{n+2} + \ldots + u_{n+p} \right| < \varepsilon, \qquad \forall\ n \geq m \text{ and } p \geq 1.$$

By Cauchy's General Principle of Convergence (for sequences), the sequence $\{S_n\}$of partial sums of $\sum u_n$ converges iff to each $\varepsilon > 0 \exists$ a positive integer *m*, such that

$$\left| S_{n+p} - S_n \right| < \varepsilon, \qquad \forall\ n \geq m \text{ and } p \geq 1$$

or

$$\left| u_{n+1} + u_{n+2} + \ldots + u_{n+p} \right| < \varepsilon, \qquad \forall\ n \geq m \text{ and } p \geq 1$$

Example 4.2. Show that the series $\sum \frac{1}{n}$ does not converge.

Solution. Suppose, if possible, the series converges.

Therefore, for any given $\varepsilon\left(\text{say}, \frac{1}{4}\right)$, $\exists$ a positive integer m, such that

$$\left|\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{n+p}\right|<\varepsilon, \quad \forall\ n \geq m \text{ and } p \geq 1$$

In particular, if $n = m$ and $p = m$, we get

$$\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{n+p}=\frac{1}{m+1}+\frac{1}{m+2}+\ldots+\frac{1}{2m}$$

$$> m \cdot \frac{1}{2m}=\frac{1}{2}>\varepsilon .$$

Thus, there is a contradiction. Hence, the given series does not converge.

It may also be seen that $\lim u_n = \lim \frac{1}{n} = 0$ even though the series does not converge.

4.1.4 Some Preliminary Theorems

Theorem 4.3. *If* $\sum u_n = u$, *then* $\sum cu_n = cu$, *where c is any number independent of n.*

The result follows at once from the identity

$$\sum_{r=1}^{n} cu_r = c \sum_{r=1}^{n} u_r$$

on making n tends to infinity.

Theorem 4.4. *If* $\sum_{n=1}^{\infty} u_n = u$, *then* $\sum_{n=0}^{\infty} u_n = u + u_0$, *and* $\sum_{n=2}^{\infty} u_n = u - u_1$.

Let $S'_n = \sum_{r=0}^{n} u_r$ and $S_n = \sum_{r=1}^{n} u_r$.

Clearly

$$S'_n = u_0 + S_n$$

$\therefore$ By letting n tends to infinity

$$\sum_{n=0}^{\infty} u_n = u_0 + u$$

The proof of the second part is similar.

A slight modification and extension enables us to conclude that the insertion or removal of any finite number of terms from a convergent series does not affect its convergence. Of course, the sums of the various series are related in the expected way.

It is also clear that if the series $\sum_{n=1}^{\infty} u_n$ is divergent, the changed series $\sum_{n=1}^{\infty} cu_n$, $\sum_{n=0}^{\infty} u_n$ or $\sum_{n=2}^{\infty} u_n$ is also divergent.

Hence, the behaviour of a series as regards convergence is not altered by

(*i*) *the alteration, addition or omission of a finite number of terms; or*

(*ii*) *multiplication of all the terms by a finite number other than zero.*

Theorem 4.5. (*Convergent series may be added or subtracted term by term*). *If* $\sum u_n = u$ *and* $\sum v_n = v$, *then* $\sum w_n = u \pm v$, *where* $w_n = u_n \pm v_n$, *for all n.*

The result follows from the identity

$$\sum_{r=1}^{n} w_r = \sum_{r=1}^{n} (u_r \pm v_r) = \sum_{r=1}^{n} u_r \pm \sum_{r=1}^{n} v_r$$

by making *n* tend to infinity.

The same proof shows that

(*i*) if any two of the three series are convergent, the third is also convergent,

(*ii*) if one of the series is divergent and another convergent then the third is necessarily divergent, but

(*iii*) if two of the series are divergent, no conclusion can be drawn about the behaviour of the third, which may converge or diverge.

Theorem 4.6. *If a series* $\sum u_n$ *converges to the sum u then so does any series obtained from* $\sum u_n$ *by grouping the terms in brackets without altering the order of the terms.*

Suppose that the series derived from $\sum u_n$ by the insertion of brackets is $\sum v_n$ and let σ_r denote the *r*th partial sum of the series $\sum v_n$. Suppose that σ_r contains n_r terms of the given series then, since the order of the terms is unaltered, $\sigma_r = S_{n_r}$.

Also as $r \to \infty$, $n_r \to \infty$.

Since the given series converges to *u*, the sequence $\{S_n\}$ of its partial sums also converges to *u*.

Hence, as $r \to \infty$ or $n_r \to \infty$, $s_{n_r} \to u$, and hence $\sigma_r \to u$.

Remarks 1. Converse of the theorem is not always true.

For example, the series $(1 - 1) + (1 - 1) + \ldots$ is convergent, whereas the series $1 - 1 + 1 - 1 + \ldots$, obtained by removing the brackets is not.

Hence, in convergent series brackets may be *inserted* at will without affecting convergence but may not be removed.

In the case of convergent positive term series, or the absolutely convergent series, however, it will be shown later that the brackets may be inserted or removed without affecting convergence.

2. The theorem may as well be proved by the following alternative argument.

Since the given series converges to *u*, its sequence $\{S_n\}$ of partial sums will also converge to *u*. Therefore, the sequence $\{\sigma_n\}$ of partial sums of $\sum v_n$, being a subsequence of $\{S_n\}$ will also converge to *u*.

3. The theorem may be restated, "*A series obtained from a given convergent series by a grouping of terms converges to the same limit*".

By *grouping* we simply mean the placing of brackets or associating the terms of the series without changing the order of the terms.

4.2 POSITIVE TERM SERIES

Series with non-negative terms are the simplest and the most important type of series one comes across. The simplicity arises mainly from the fact that the sequence of its partial sums is monotonic increasing.

Let $\sum u_n$ be an infinite series of positive terms and $\{S_n\}$, the sequence of its partial sums, so that

$$S_n = u_1 + u_2 + ... + u_n \geq 0, \quad \forall\ n$$

$$\therefore \qquad S_n - S_{n-1} = u_n \geq 0$$

$$\Rightarrow \qquad S_n \geq S_{n-1},$$

Thus the sequence $\{S_n\}$ of partial sums of a series of positive terms is a monotonic increasing sequence.

Since a monotonic increasing sequence can either converge, or diverge to ∞, but cannot oscillate, therefore, there are only two possibilities for a positive term series—it may either converge, or diverge to $+\infty$.

Theorem 4.7. *A positive term series converges iff the sequence of its partial sums is bounded above.*

We know that the sequence of partial sums of a positive term series is a monotonic increasing sequence and a monotonic increasing sequence converges iff it is bounded above. Therefore, it follows that a positive term series converges iff its sequence of partial sums is bounded above.

Remarks 1. The sequence of partial sums of a series with negative terms can be shown to be monotonic decreasing and hence a series with negative terms converges iff the sequence of its partial sums is bounded below.

2. It may similarly be seen that a series of negative terms can either converge, or diverge to $-\infty$.

3. A series of positive terms can either converge, or diverge to $+\infty$. But a series with arbitrary terms can have five possible behaviours depending upon the behaviour of the sequence of its partial sums.

4.2.1 Geometric Series

The positive term geometric series $1 + r + r^2 + ...$, converges for $r < 1$, and diverges to $+\infty$ for $r \geq 1$.

Case I. $0 \leq r < 1$.

Let $\{S_n\}$ be the sequence of its partial sums, so that

$$S_n = 1 + r + r^2 + ... + r^{n-1}$$

i.e.,

$$S_n = \frac{1 - r^n}{1 - r} = \frac{1}{1 - r} - \frac{r^n}{1 - r} \leq \frac{1}{1 - r}, \quad \forall\ n,$$

so that $\{S_n\}$ is bounded above and hence the series converges.

Case II. When $r = 1$, $S_n = n$, so that $\{S_n\}$ is not bounded above and hence the series diverges to $+\infty$.

Case III. When $r > 1$, every term of S_n after the first is greater than 1, so that

$$S_n > n, \quad \forall\ n.$$

Therefore, the sequence $\{S_n\}$ is not bounded above and consequently the given series diverges to $+\infty$.

Hence, the given series converges if $r < 1$ and diverges if $r \geq 1$.

4.2.2 A Comparison Series

As mentioned earlier, an important technique for testing the convergence of a series is to compare the given series with a suitably selected series with known behaviour. We now discuss one such series which is most frequently used for such a purpose.

Theorem 4.8. *A positive term series* $\Sigma\frac{1}{n^p}$ *is convergent if and only if* $p > 1$.

Let S_n denote the sum of the first n terms of the given series.

Case I. When $p > 1$.

Now

$$\frac{1}{1^p} = 1$$

$$\frac{1}{2^p} + \frac{1}{3^p} < \frac{2}{2^p} = \frac{1}{2^{p-1}}$$

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{4}{4^p} = \frac{1}{4^{p-1}} = \left(\frac{1}{2^{p-1}}\right)^2$$

$$\frac{1}{8^p} + \frac{1}{9^p} + \ldots + \frac{1}{15^p} < \frac{8}{8^p} = \frac{1}{8^{p-1}} = \left(\frac{1}{2^{p-1}}\right)^3$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{1}{\left(2^n\right)^p} + \frac{1}{\left(2^n + 1\right)^p} + \ldots + \frac{1}{\left(2^{n+1} - 1\right)^p} < \frac{2^{n+1} - 2^n}{\left(2^n\right)^p} = \frac{1}{\left(2^n\right)^{p-1}} = \left(\frac{1}{2^{p-1}}\right)^n$$

Adding,

$$S_{2^{n+1}-1} < 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \ldots + \left(\frac{1}{2^{p-1}}\right)^n$$

$$= \frac{1 - \left(\frac{1}{2^{p-1}}\right)^{n+1}}{1 - \frac{1}{2^{p-1}}} = \frac{2^{p-1}\left[1 - \left(\frac{1}{2^{p-1}}\right)^{n+1}\right]}{2^{p-1} - 1}$$

$$< \frac{2^{p-1}}{2^{p-1} - 1}, \quad \forall\ n$$

We know that when n is any positive integer,

$$2^{n+1} - 1 > 2^n > n$$

$$\therefore \qquad S_n < S_{2^n} < S_{2^{n+1}-1} < \frac{2^{p-1}}{2^{p-1} - 1}.$$

Since for a given p, $\frac{2^{p-1}}{2^{p-1}-1}$ is a fixed number, therefore, the sequence $\{S_n\}$ of partial sums of the given positive term series is bounded above and hence the series converges for $p > 1$.

Case II. When $p \leq 1$.

We know, if n is any positive integer and $p \leq 1$, then

$$n^p \leq n \Rightarrow \frac{1}{n^p} \geq \frac{1}{n}$$

$$\therefore \quad 1 + \frac{1}{2^p} \geq 1 + \frac{1}{2} > \frac{1}{2}$$

$$\frac{1}{3^p} + \frac{1}{4^p} \geq \frac{1}{3} + \frac{1}{4} > \frac{2}{4} = \frac{1}{2}$$

$$\frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \frac{1}{8^p} \geq \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{4}{8} = \frac{1}{2}$$

$$\frac{1}{9^p} + \frac{1}{10^p} + \ldots + \frac{1}{16^p} \geq \frac{1}{9} + \frac{1}{10} + \ldots + \frac{1}{16} > \frac{8}{16} = \frac{1}{2}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{1}{\left(2^{m-1}+1\right)^p} + \frac{1}{\left(2^{m-1}+2\right)^p} + \ldots + \frac{2}{\left(2^m\right)^p} \geq \frac{1}{2^{m-1}+1} + \frac{1}{2^{m-1}+2} + \ldots + \frac{1}{2^m} > \frac{2^m - 2^{m-1}}{2^m} = \frac{1}{2}$$

Adding,

$$S_{2^m} > \frac{m}{2}.$$

We shall now show that $\{S_n\}$ is not bounded above.

If G be any number, however large, then $\exists\, m \in \mathbf{N}$ such that

$$\frac{m}{2} > G$$

Let $n > 2^m$,

$$\therefore \quad S_n > S_{2^m} > G$$

Thus, the sequence $\{S_n\}$ of partial sums of the given series is not bounded above, and hence the series diverges for $p \leq 1$.

Thus, the given series $\Sigma \frac{1}{n^p}$ converges iff $p > 1$.

4.3 COMPARISON TESTS FOR POSITIVE TERM SERIES

Two types of comparison tests shall now be discussed. In the first type, the general term of one series will be compared with the general term of the second series. In the second type, the ratio of two consecutive

terms of one series will be compared to the ratio of the corresponding consecutive terms of the second series.

4.3.1 Comparison Test (*First type*)

I. *If* $\sum u_n$ *and* $\sum v_n$ *are two positive term series, and* $k \neq 0$, *a fixed positive real number (independent of n) and there exists a positive integer m such that* $u_n \leq kv_n$, $\forall\ n \geq m$, *then*

(*i*) $\sum u_n$ *is convergent, if* $\sum v_n$ *is convergent, and*

(*ii*) $\sum v_n$ *is divergent, if* $\sum u_n$ *is divergent.*

Let $n \geq m$, $S_n = u_1 + u_2 + ... + u_n$, and $t_n = v_1 + v_2 + ... + v_n$.

Now $\forall\ n \geq m$, we have

$$S_n - S_m = u_{m+1} + u_{m+2} + ... + u_n$$

$$\leq k\left(v_{m+1} + v_{m+2} + ... + v_n\right) = k\left(t_n - t_m\right)$$

or $$S_n \leq kt_n + \left(S_m - kt_m\right)$$

$\Rightarrow$ $$S_n \leq kt_n + h \qquad ...(1)$$

where $h = S_m - kt_m$, is a finite quantity.

(*i*) If $\sum v_n$ is convergent, then the sequence $\{t_n\}$ of its partial sums is bounded above, so that $\exists$ a number B such that

$$t_n \leq B,\ \forall\ n.$$

So from (1), we get

$S_n \leq kB + h$, for all $n \geq m$,

$\Rightarrow$ the sequence $\{S_n\}$ is bounded above.

$\Rightarrow$ $\sum u_n$ is convergent.

(*ii*) If $\sum u_n$ is divergent, then the sequence $\{S_n\}$ of its partial sums is not bounded above, so that if G be any number, however, large, $\exists$ a positive integer m_0 such that

$$S_n > G \quad \forall\ n \geq m_0.$$

Thus from (1), $\forall\ n \geq \max\ (m, m_0)$,

$$t_n \geq \frac{1}{k}(G - h),\ k \neq 0$$

$\Rightarrow$ the sequence $\{t_n\}$ is unbounded

$\Rightarrow$ $\sum v_n$ is divergent.

II. Limit Form. *If* $\sum u_n$ *and* $\sum v_n$ *are two positive term series such that* $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = l$, *where l is a non-zero finite number, then the two series converge or diverge together.*

Evidently $l > 0$.

Let ε be a positive number such that $l - \varepsilon > 0$.

Since $\lim_{n\to\infty} \frac{u_n}{v_n} = l$, therefore $\exists$ a positive integer m such that

$$\left|\frac{u_n}{v_n} - l\right| < \varepsilon, \qquad \forall\ n \geq m$$

$$\Rightarrow \qquad l - \varepsilon < \frac{u_n}{v_n} < 1 + \varepsilon, \qquad \forall\ n \geq m$$

$$\Rightarrow \qquad (l - \varepsilon)v_n < u_n < (1 + \varepsilon)v_n, \qquad \forall\ n \geq m \qquad \ldots(1)$$

Now, if Σv_n is convergent, then from (1)

$$u_n < (1 + \varepsilon)v_n, \qquad \forall\ n \geq m$$

so that by Test I, Σu_n is convergent.

Again, if Σv_n is divergent, then from (1)

$$u_n > (1 - \varepsilon)v_n, \qquad \forall\ n \geq m$$

so that by Test I, Σu_n is divergent.

Similarly, we may show that Σv_n converges or diverges with Σu_n. Hence, the two series behave alike.

4.3.2 Comparison Test (*Second type*)

III. *If Σu_n and Σv_n are two positive term series and $\exists$ a positive integer m such that*

$$\frac{u_n}{u_{n+1}} \geq \frac{v_n}{v_{n+1}}, \quad \forall\ n \geq m,$$

then (i) Σu_n is convergent, if Σv_n is convergent, and (ii) Σv_n is divergent, if Σu_n is divergent.

Let m be a positive integer and $S_n = u_1 + u_2 + \ldots + u_n$

and $t_n = v_1 + v_2 + \ldots + v_n$.

For $n \geq m$, we have

$$\frac{u_m}{u_n} = \frac{u_m}{u_{m+1}} \cdot \frac{u_{m+1}}{u_{m+2}} \ldots \frac{u_{n-1}}{u_n} \geq \frac{v_m}{v_{m+1}} \cdot \frac{v_{m+1}}{v_{m+2}} \ldots \frac{v_{n-1}}{v_n} = \frac{v_m}{v_n}$$

$$\Rightarrow \qquad u_n \leq \frac{u_m}{v_m} v_n.$$

Since m is a fixed positive integer, therefore u_m/v_m is fixed number, say k. Thus $\forall\ n \geq m$, we have

$$u_n \leq k v_n$$

Hence by Test I, Σu_n converges if Σv_n converges and Σv_n diverges if Σu_n diverges.

Notes 1. For practical purposes, Test II is very useful and can be easily applied.

2. For a successful application of the comparison test, we first make an estimate of the magnitude of the general term u_n of the given series, and then select the auxiliary series $\sum v_n$ of such a magnitude that $\lim(u_n/v_n) = l \neq 0, \infty,$ or in other words $u_n \sim v_n$. Thus for large values of n,

$$\sqrt{n^3+1} \pm \sim \pm n^{3/2}, \frac{n^r}{(1+n)^s} \sim \pm n^{r-s}$$

$$\sin\frac{1}{n} \pm \sim \pm \frac{1}{n}$$

3. It will help to remember that for large n, $e^{an} \gg n^b \gg (\log n)^c$, where a, b, c are positive numbers.

4.3.3 Solved Examples

Example 4.3. Show that the series $1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$ is convergent.

Solution. We have

$$\frac{1}{2!} = \frac{1}{2}$$

$$\frac{1}{3!} < \frac{1}{2^2}$$

$$\frac{1}{4!} < \frac{1}{2^3}$$

$$\ldots \quad \ldots$$

$$\ldots \quad \ldots$$

$$\frac{1}{n!} < \frac{1}{2^{n-1}}$$

$$\therefore \qquad 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \ldots < 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots$$

Thus each term of the given series after the second is less than the corresponding term of the convergent geometric series

$$1 + \frac{1}{2} + \frac{1}{2^2} + \ldots$$

Thus by Test I, the given series converges.

Example 4.4. Show that the series

$$\frac{1}{(\log 2)^p} + \frac{1}{(\log 3)^p} + \ldots + \frac{1}{(\log n)^p} + \ldots$$

diverges for $p > 0$.

Solution. Since $\lim_{n\to\infty} \frac{(\log n)^p}{n} = 0,$ therefore

$$(\log n)^p < n, \quad \forall\ n > 1$$

$$\therefore \qquad \frac{1}{(\log n)^p} > \frac{1}{n}, \quad \forall\ n > 1$$

Let us compare the given series with the divergent series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

Since each term of the given series is greater than the corresponding term of the divergent series, therefore, the given series diverges.

Example 4.5. Show that the series

$$\frac{1 \cdot 2}{3^2 \cdot 4^2} + \frac{3 \cdot 4}{5^2 \cdot 6^2} + \frac{5 \cdot 6}{7^2 \cdot 8^2} + \ldots, \text{ is convergent}$$

Solution. Let us denote the given series by $\sum u_n,$ where

$$u_n = \frac{(2n-1)(2n)}{(2n+1)^2 (2n+2)^2}, \left(\sim \frac{1}{n^2}\right).$$

Let us compare it with the convergent series $\sum v_n,$

where $$v_n = 1/n^2.$$

Now $$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{(2+1/n)\ (2+2/n)}{(2+3/n)(2+4/n)} = 1$$

Thus, the two series converge or diverge together.

Since, $\sum v_n$ converges, therefore $\sum u_n$ also converges.

Example 4.6. Investigate the behaviour of the series whose nth term is $\sin 1/n$.

Solution. Let $u_n = \sin 1/n$, and $v_n = 1/n$.

Now

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sin 1/n}{1/n} = 1$$

Therefore, the two series behave alike.

Since, $\sum v_n$ diverges, therefore $\sum \sin 1/n$ also diverges.

Example 4.7. Test for convergence of the series whose nth term is

$$\{(n^3 + 1)^{1/3} - n\}$$

Solution. Let $u_n = \left(n^3 + 1\right)^{1/3} - n$

$$= n\left\{\left(1 + \frac{1}{n^3}\right)^{1/3} - 1\right\}$$

$$= n\left\{\frac{1}{3n^3} + \ldots\right\} = \frac{1}{3n^2} + \ldots \left(\sim \frac{1}{n^2}\right)$$

Let $\sum v_n = \sum \frac{1}{n^2}$

Now

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{3}$$

Therefore, the two series converge and diverge together.

Since, $\sum v_n$ converges, therefore, the given series also converges.

Example 4.8. Test the convergence of the series $\sum \frac{1}{n^{1+1/n}}$.

Solution. Let $u_n = \sum \frac{1}{n^{1+1/n}}$ and $v_n = \frac{1}{n}$

Now $$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = 1$$

Hence, the two series $\sum u_n$ and $\sum v_n$ behave alike.

Since, $\sum v_n$ is divergent, therefore $\sum \frac{1}{n^{1+1/n}}$ is also divergent.

EXERCISE

1. $\frac{1}{1\cdot 2\cdot 3} + \frac{3}{2\cdot 3\cdot 4} + \frac{5}{3\cdot 4\cdot 5} + \frac{7}{4\cdot 5\cdot 6} + \ldots$
2. $\frac{1}{\sqrt{1\cdot 2}} + \frac{1}{\sqrt{2\cdot 3}} + \frac{1}{\sqrt{3\cdot 4}} + \ldots$
3. $\frac{1}{4\cdot 6} + \frac{\sqrt{3}}{6\cdot 8} + \frac{\sqrt{5}}{8\cdot 10} + \frac{\sqrt{7}}{10\cdot 12} + \ldots$
4. $\sum \frac{n+1}{n^p}$
5. $\sum \frac{1}{\sqrt{n} + \sqrt{n+1}}$
6. $\sum\left(\sqrt{n^4+1} - \sqrt{n^4-1}\right)$ [*Hint*: Rationalize]
7. (*i*) $\sum \sin \frac{1}{n^2}$, (*ii*) $\sum \cos \frac{1}{n}$

8. $\Sigma \frac{1}{\sqrt{n}} \tan \frac{1}{n}$

9. $\Sigma \left\{ \frac{\sqrt{n+1} - \sqrt{n-1}}{n} \right\}$

10. Show that the series $1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \ldots$ converges.

[*Hint*: $n^n > 2^n$, for $n > 2$. Compare with the convergent geometric series $\Sigma 1/2^n$]

ANSWERS

1. Convergent, 2. Divergent, 3. Convergent, 4. Convergent for $p > 2$, 5. Divergent, 6. Convergent, 7. (*i*) Convergent, (*ii*) Divergent, 8. Convergent, 9. Convergent.

4.4 CAUCHY'S ROOT TEST

If Σu_n *is a positive term series, such that* $\lim_{n\to\infty} (u_n)^{1/n} = l$, *then the series*

(*i*) *converges, if* $l < 1$,

(*ii*) *diverges, if* $l > 1$, *and*

(*iii*) *the test fails to give any definite information, if* $l = 1$.

Case I. $l < 1$.

Let us select a positive number ε, such that $l + \varepsilon < 1$.

Let $l + \varepsilon = \alpha < 1$.

Since, $\lim_{n\to\infty} (u_n)^{1/n} = l$, therefore $\exists$ a positive integer m such that

$$\left| (u_n)^{1/n} - l \right| < \varepsilon, \qquad \forall\ n \geq m$$

$$\Rightarrow \quad l - \varepsilon < (u_n)^{1/n} < l + \varepsilon, \qquad \forall\ n \geq m$$

$$\Rightarrow \quad (l - \varepsilon)^n < u_n < (l + \varepsilon)^n = \alpha^n, \qquad \forall\ n \geq m$$

$$\Rightarrow \quad u_n < \alpha^n, \qquad \forall\ n \geq m.$$

But since $\Sigma \alpha^n$ is a convergent geometric series (common ratio $\alpha < 1$), therefore, by comparison test, the series Σu_n converges.

Case II. $l > 1$.

Let us select a positive number ε such that $l - \varepsilon > 1$

Let $l - \varepsilon = \beta > 1$.

Since, $\lim_{n\to\infty} (u_n)^{1/n} = l$, therefore $\exists$ a positive integer m_1 such that

$$l - \varepsilon < (u_n)^{1/n} < l + \varepsilon, \qquad \forall\ n \geq m_1$$

$$\Rightarrow (l - \varepsilon)^n < u_n < (l - \varepsilon)^n, \qquad \forall\ n \geq m_1$$

$$\Rightarrow u_n > (l - \varepsilon)^n = \beta^n, \qquad \forall\ n \geq m_1.$$

But since $\sum \beta^n$ is a divergent geometric series (common ratio $\beta > 1$), therefore by comparison test, the series $\sum u_n$ diverges.

Note. The test fails to give any definite information for $l = 1$.

Consider the two series $\sum(1/n)$ and $\sum(1/n^2)$.

$\sum(1/n)$ diverges, while $\lim\limits_{n\to\infty} (1/n)^{1/n} = 1$, and $\sum(1/n^2)$, converges, while $\lim\limits_{n\to\infty} (1/n^2)^{1/n} = 1$.

Example 4.9. Test for convergence of the series whose general term is $\left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$.

Solution. Let $u_n = \dfrac{1}{\left(1 + 1/\sqrt{n}\right)^{n^{3/2}}}$, then

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{1}{\left(1 + 1/\sqrt{n}\right)^{\sqrt{n}}} = \frac{1}{e} < 1.$$

Hence, the series converges.

4.5 D'ALEMBERT'S RATIO TEST

If $\sum u_n$ *is a positive term series, such that* $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = l$, *then the series*

(*i*) *converges, if* $l < 1$,

(*ii*) *diverges, if* $l > 1$, *and*

(*iii*) *the test fails, if* $l = 1$.

Case I. $0 < l < 1$.

Let us select a positive number ε, such that $l + \varepsilon < 1$.

Let $l + \varepsilon = \alpha < 1, \alpha \neq 0$.

Since, $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = l$, therefore $\exists$ a positive integer m such that

$$\left| \frac{u_{n+1}}{u_n} - l \right| < \varepsilon, \quad \forall\ n \geq m$$

$$\Rightarrow \qquad l - \varepsilon < \frac{u_{n+1}}{u_n} < l + \varepsilon, \qquad \forall\ n \geq m$$

$$\Rightarrow \qquad \frac{u_{n+1}}{u_n} < l + \varepsilon = \alpha, \qquad \forall\ n \geq m$$

For $n \geq m$,

$$\frac{u_n}{u_m} = \frac{u_{m+1}}{u_m} \cdot \frac{u_{m+2}}{u_{m+1}} \ldots \frac{u_n}{u_{n-1}} < \alpha^{n-m}$$

$$\Rightarrow \qquad u_n < \left(\frac{u_m}{\alpha^m}\right)\alpha^n, \qquad \forall\ n \geq m,\ \alpha < 1.$$

Since m is a fixed integer, therefore $\left(\frac{u_m}{\alpha^m}\right)$ is a fixed finite number, say k.

Thus, $\forall\ n \geq m,$ we have

$$u_n < k\alpha^n$$

But since $\sum \alpha^n$ is a convergent geometric series (common ratio, $\alpha < 1$), therefore by comparison test $\sum u_n$ converges.

Case II. $l > 1$.

Let us select a positive number ε, such that $l - \varepsilon > 1$.

Let $l - \varepsilon = \beta > 1$.

Since, $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = l$, therefore $\exists$ a positive integer m_1 such that

$$l - \varepsilon < \frac{u_{n+1}}{u_n} < l + \varepsilon, \qquad \forall\ n \geq m_1$$

$$\Rightarrow \qquad \frac{u_{n+1}}{u_n} > l - \varepsilon = \beta, \qquad \forall\ n \geq m_1$$

Now, for $n \geq m_1$,

$$\frac{u_n}{u_{m_1}} = \frac{u_{m_1+1}}{u_{m_1}} \cdot \frac{u_{m_1+2}}{u_{m_1+1}} \ldots \frac{u_n}{u_{n-1}} \geq \beta^{n-m_1}$$

$$\Rightarrow \qquad u_n \geq \frac{u_{m_1}}{\beta^{m_1}}\beta^n, \qquad \forall\ n \geq m_1$$

Since m_1 is a fixed integer, therefore u_{m_1}/β^{m_1} is a fixed finite number, say k_1.

Thus, for $n \geq m_1$, we have

$$u_n > k_1\beta^n.$$

But $\Sigma\beta^n$ is a divergent geometric series (common ratio, $\beta > 1$); therefore by comparison test, Σu_n diverges.

Note. The test fails for $l = 1$ in the sense that it fails to give any definite information.

For example, consider the two series $\Sigma(1/n)$ and $\Sigma(1/n^2)$

$\Sigma\dfrac{1}{n}$ diverges, while $\lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty}\dfrac{n}{n+1} = 1$, and $\Sigma\dfrac{1}{n^2}$ converges, while $\lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty}\left(\dfrac{n}{n+1}\right)^2 = 1.$

Example 4.10. Test for convergence of the series $\Sigma\dfrac{n^2-1}{n^2+1}x^n$, $x > 0$.

Solution. Let $$u_n = \frac{n^2-1}{n^2+1}x^n.$$

$$\therefore \qquad \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{n^2+1}{n^2-1}\cdot\frac{(n+1)^2-1}{(n+1)^2+1}\cdot\frac{x^{n+1}}{x^n} = x$$

Hence by D'Alembert's Ratio Test the series converges if $x < 1$ and diverges if $x > 1$.

The test fails to give any information when $x = 1$.

When $x = 1$, $u_n = \dfrac{n^2-1}{n^2+1}$, and $\lim\limits_{n\to\infty} u_n = 1 \neq 0$

$\Rightarrow$ The series is divergent.

Hence, the series converges if $x < 1$ and diverges if $x \geq 1$.

4.6 RAABE'S TEST

If Σu_n is a positive term series, such that $\lim\limits_{n\to\infty} n\left(\dfrac{u_n}{u_{n+1}} - 1\right) = 1$, then the series

(*i*) *converges, if* $l > 1$,

(*ii*) *diverges, if* $l < 1$, *and*

(*iii*) *the test fails, if* $l = 1$.

Case I. $l > 1$.

Let us select a positive number ε, such that $l - \varepsilon > 1$.

Let $l - \varepsilon = \alpha > 1$.

Since $\lim\limits_{n\to\infty} n\left(\dfrac{u_n}{u_{n+1}} - 1\right) = l$, therefore $\exists$ a positive integer m such that for all $n \geq m$,

$$l - \varepsilon < n\left(\frac{u_n}{u_{n+1}} - 1\right) < l + \varepsilon$$

$$\Rightarrow \quad n\left(\frac{u_n}{u_{n+1}} - 1\right) > l - \varepsilon = \alpha$$

$$\Rightarrow \quad nu_n - nu_{n+1} > \alpha u_{n+1}$$

$$\Rightarrow \quad nu_n - (n+1)u_{n+1} > (\alpha - 1)u_{n+1}, \quad \alpha - 1 > 0$$

Putting $n = m, m + 1, m + 2, ..., n - 1$, and adding, we get

$$mu_m - nu_n > (\alpha - 1)\left(u_{m+1} + u_{m+2} + ... + u_n\right)$$

$$= (\alpha - 1)(S_n - S_m), \text{ where } S_n = \sum_{r=1}^{n} u_r$$

$$\Rightarrow \quad (\alpha - 1)(S_n - S_m) < mu_m, \qquad \forall\ n \geq m$$

$$\Rightarrow \quad S_n < S_m + \frac{m}{\alpha - 1} u_m, \quad \forall\ n \geq m,$$

Since m is a fixed integer, therefore $S_m + \dfrac{m}{\alpha - 1} u_m$ is a fixed finite number.

Thus, the sequence $\{S_n\}$ of partial sums of the given series is bounded above and hence the series $\sum u_n$ is convergent.

Case II. $l < 1$.

Let us select a positive number ε, such that $l + \varepsilon < 1$.

Since $\lim\limits_{n\to\infty} n\left(\dfrac{u_n}{u_{n+1}} - 1\right) = l$, therefore $\exists$ a positive integer m such that for all $n \geq m$,

$$l - \varepsilon < n\left(\frac{u_n}{u_{n+1}} - 1\right) < l + \varepsilon < 1$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} < \frac{n+1}{n}, \quad \forall\ n \geq m$$

If $v_n = 1/n$, the series $\sum v_n$ is divergent, and,

$$\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$$

Hence by comparison test the series $\sum u_n$ diverges.

Notes 1. The test fails to give any definite information for $l = 1$. Consider the two series $\Sigma\frac{1}{n}$ and $\Sigma\frac{1}{n(\log n)^2}$.

The former is divergent, while the later is convergent but

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = 1 \text{ for both.}$$

2. Raabe's Test is stronger that D'Alembert's Ratio Test and may succeed where Ratio Test fails.

Example 4.11. Test for convergence of the series

$$\frac{\alpha}{\beta} + \frac{1+\alpha}{1+\beta} + \frac{(1+\alpha)(2+\alpha)}{(1+\beta)(2+\beta)} + \ldots$$

Solution. Here

$$u_n = \frac{(1+\alpha)(2+\alpha)(3+\alpha)\ldots(n-1+\alpha)}{(1+\beta)(2+\beta)(3+\beta)\ldots(n-1+\beta)}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{n+\alpha}{n+\beta} = 1$$

Hence, the Ratio Test fails.

Again

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{n+\beta}{n+\alpha} - 1\right) = \lim_{n\to\infty}\frac{\beta-\alpha}{1+\alpha/n} = \beta - \alpha$$

Thus by Raabe's Test, the series converges if $\beta - \alpha > 1$ or $\beta > \alpha + 1$, and diverges if $\beta < \alpha + 1$. The test fails for $\beta = \alpha + 1$.

But for $\beta = \alpha + 1$, the series becomes

$$\frac{\alpha}{\alpha+1} + \frac{1+\alpha}{2+\alpha} + \frac{1+\alpha}{3+\alpha} + \ldots = \Sigma\frac{1+\alpha}{n+\alpha}$$

which diverges, by comparison with $\Sigma 1/n$.

Example 4.12. Show that the series

$$\Sigma\frac{3\cdot6\cdot9\ldots3n}{7\cdot10\cdot13\ldots(3n+4)}x^n, \quad x > 0$$

converges for $x \le 1$, and diverges for $x > 1$.

Solution. Now

$$\frac{u_n}{u_{n+1}} = x^n\frac{3\cdot6\cdot9\ldots3n}{7\cdot10\ldots(3n+4)}\cdot\frac{7\cdot10\cdot13\ldots(3n+4)(3n+7)}{3\cdot6\cdot9\ldots3n(3n+3)}\frac{1}{x^{n+1}}$$

$$= \frac{3n+7}{3n+3} \cdot \frac{1}{x}$$

$$\therefore \qquad \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x \ .$$

By Ratio Test, the series converges for $x < 1$ and diverges for $x > 1$. The test fails for $x = 1$. But for $x = 1$,

$$\frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3} \ .$$

$$\therefore \qquad \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{4n}{3n+3} = \frac{4}{3} > 1$$

$\therefore$ By Raabe's Test, the series converges.

Thus the series converges for $0 < x \leq 1$ and diverges for $x > 1$.

EXERCISE

Test the behaviour of the following series:

1. $\frac{1}{2} + \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} + \frac{1}{4\cdot 2^4} + \ldots$

2. $\left(\frac{1}{3}\right)^2 + \left(\frac{1\cdot 2}{3\cdot 5}\right)^2 + \left(\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)^2 + \ldots$

3. $\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots,$

4. $\frac{x^2}{2\sqrt{1}} + \frac{x^3}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^5}{5\sqrt{4}} + \ldots,$

5. $1 + \frac{x^2}{2^p} + \frac{x^4}{4^p} + \frac{x^6}{6^p} + \ldots$

6. $\frac{1^2\cdot 2^2}{1!} + \frac{2^2\cdot 3^2}{2!} + \frac{3^2\cdot 4^2}{3!} + \frac{4^2\cdot 5^2}{4!} + \ldots$

7. $\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{3^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \ldots$

8. $\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \ldots$

Test for convergence each of the following series whose nth terms are given:

9. $\dfrac{\lfloor n}{n^n}, r > 0$

10. $\dfrac{1 \cdot 2 \cdot 3 \ldots n}{7 \cdot 10 \ldots (3n+4)}$

11. $\dfrac{r^n}{n^n}, \quad r > 0$

12. $\dfrac{\sqrt{n}\, x^n}{\sqrt{n^2+1}}, \quad x > 0$

13. $\dfrac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots 2n} \cdot \dfrac{1}{n}$

14. $\dfrac{1 \cdot 3 \cdot 5 \ldots (4n-3)}{2 \cdot 4 \cdot 6 \ldots (4n-2)} \cdot \dfrac{x^{2n}}{4n}$

15. $\sqrt{\dfrac{n-1}{n^3+1}}\, x^n, \quad x > 0$

16. $\dfrac{2 \cdot 4 \cdot 6 \ldots (2n+2)}{3 \cdot 5 \cdot 7 \ldots (2n+3)}\, x^{n-1}$

ANSWERS

1. Converges
2. Converges
3. Converges for $x < 1$, diverges for $x \geq 1$
4. Converges for $x \leq 1$, diverges for $x > 1$
5. Converges for $|x| < 1$, diverges for $|x| > 1$; for $x = 1$, converges for $p > 1$, diverges for $p \leq 1$
6. Convergent
7. Convergent
8. Convergent
9. Convergent
10. Convergent
11. Convergent
12. Convergent for $x < 1$, divergent for $x \geq 1$
13. Convergent
14. Convergent for $|x| \leq 1$, divergent for $|x| > 1$
15. Convergent for $x < 1$, divergent for $x \geq 1$
16. Convergent for $x < 1$, divergent for $x \geq 1$.

4.7 LOGARITHMIC TEST

If $\sum u_n$ *is positive term series such that,*

$$\lim_{n \to \infty} \left(n \log \frac{u_n}{u_{n+1}} \right) = l,$$

then the series converges for $l > 1$, *and diverges for* $l < 1$.

First, let $l > 1$.

Let us select $\varepsilon > 0$, such that $l - \varepsilon < 1$.

Let $l - \varepsilon = \alpha > 1$.

Since $\lim_{n\to\infty}\left(n \log \frac{u_n}{u_{n+1}}\right) = l$, therefore $\exists$ an m, such that

$$l - \varepsilon < n \log \frac{u_n}{u_{n+1}} < l + \varepsilon, \quad \forall\ n \geq m$$

$$\Rightarrow \qquad n \log \frac{u_n}{u_{n+1}} > \alpha, \quad \forall\ n \geq m$$

$$\Rightarrow \qquad \frac{u_n}{u_{n+1}} > e^{\alpha/n}, \quad \forall\ n \geq m$$

Now since $\{(1 + 1/n)^n\}$ is a monotonic increasing sequence converging to e, therefore

$$\left(1 + \frac{1}{n}\right)^n \leq e, \quad \forall\ n$$

so that, we get

$$\frac{u_n}{u_{n+1}} > \left(1 + \frac{1}{n}\right)^{\alpha} = \frac{(n+1)^{\alpha}}{n^{\alpha}} = \frac{v_n}{v_{n+1}}, \quad \forall\ n \geq m$$

where $v_n = 1/n^{\alpha}$.

But since for $\alpha > 1$, $\sum v_n$ converges, therefore by comparison test, $\sum u_n$ also converges.

We may similarly show that for $l < 1$, the series $\sum u_n$ diverges.

Example 4.13. Test for convergence the series

$$1 + \frac{x}{1!} + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \ldots, \text{ for } x > 0$$

Solution. Ignoring the first term, $u_n = \frac{n^n x^n}{n!}$.

Now $$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n x = ex$$

By Ratio Test, the series converges for $ex < 1$ or $x < 1/e$, and diverges for $x > 1/e$.

For $x = 1/e$,

$$\frac{u_n}{u_{n+1}} = \left(\frac{n}{n+1}\right)^n \cdot e$$

Now

$$\lim_{n\to\infty}\left(n \log \frac{u_n}{u_{n+1}}\right) = \lim_{n\to\infty} n\left[1 - n \log\left(1 + \frac{1}{n}\right)\right].$$

$$= \lim_{n\to\infty} n\left[1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\right)\right] = \frac{1}{2} < 1$$

Therefore by logarithmic test, the series diverges.

Hence, the series converges for $x < 1/e$, and diverges for $x \geq 1/e$.

Note. Logarithmic test is generally more helpful in situations like those above, where the presence of a number like e in u_n/u_{n+1} makes the application of Raabe's Test difficult.

4.8 INTEGRAL TEST

Improper Integral. As preparatory to the introduction of Cauchy's Integral Test, it will help to remember that the infinite integral $\int_a^\infty u(x)\,dx$ is said to converge if $t(x) = \int_a^x u(x)\,dx$ tends to a finite limit as $x \to \infty$; otherwise the integral is said to diverge.

Further, if $u(x) \geq 0$ for all $x > a$, it can be shown geometrically or otherwise that the integral $\int_a^t u(x)\,dx$ is a monotonic increasing function of t, so that the improper integral

$$\int_a^\infty u(x)\,dx, \text{ where } u(x) \geq 0, \ \forall\ x > a$$

converges iff it is bounded above *i.e.*, $\exists$ a positive number k such that

$$\int_a^t u(t)\,dt \leq k, \ \forall\ t \geq a$$

4.8.1 Cauchy's Integral Test

If u is a non-negative monotonic decreasing integrable function such that $u(n) = u_n$, for all positive integral values of n, then the series $\sum_{n=1}^{\infty} u_n$ and $\int_1^\infty u(x)\,dx$ converge or diverge together.

As u is monotonic decreasing, we have

$$u(n) \geq u(x) \geq u(n+1), \text{ whenever } n \leq x \leq n+1$$

Also, since u is non-negative and integrable,

$$\int_n^{n+1} u(n)\,dx \geq \int_n^{n+1} u(x)\,dx \geq \int_n^{n+1} u(n+1)\,dx$$

$$\Rightarrow \qquad u(n) \geq \int_n^{n+1} u(x)\,dx \geq u(n+1)$$

or
$$u_n \geq \int_n^{n+1} u(x)\,dx \geq u_{n+1}. \qquad \ldots(1)$$

Let us write $S_n = u_1 + u_2 + \ldots + u_n$, and $I_n = \int_1^n u(x)\,dx$, and putting $n = 1, 2, \ldots, (n-1)$ successively, and adding, we get

$$S_n - u_n \geq I_n \geq S_n - u_1$$

$$\Rightarrow \qquad 0 < u_n \leq S_n - I_n \leq u_1 \qquad \ldots(2)$$

Let us consider the sequence $\{(S_n - I_n)\}$.

$$\begin{aligned}(S_n - I_n) - (S_{n-1} - I_{n-1}) &= S_n - S_{n-1} - (I_n - I_{n-1}) \\ &= u_n - \int_{n-1}^{n} u(x)\,dx \\ &\leq 0 \qquad \text{[using (1)]}.\end{aligned}$$

Therefore, the sequence $\{(S_n - I_n)\}$ is monotonic decreasing, bounded by 0 and u_1.

Hence, the sequence converges and has a limit such that

$$0 \leq \lim (S_n - I_n) \leq u_1 \qquad \ldots(3)$$

Thus the series $\sum u_n$ converges or diverges with the integral $\int_1^{\infty} u(x)\,dx$; if convergent, the sum of the series differs from the integral by less than u_1; if divergent, the limit of $(S_n - I_n)$ still exists and lies between 0 and u_1.

Example 4.14. Show that the series $\sum(1/n^p)$ converges, if $p > 1$, and diverges if $p \leq 1$.

Solution. Let $u(x) = 1/x^p$, so that for $x \geq 1$, the function u is a non-negative monotonic decreasing integrable function such that

$$u_n = u(n) = \frac{1}{n^p}, \quad \forall\ n \in \mathbf{N}$$

By Integral Test, $\sum_{n=1}^{\infty} u_n$ and $\int_1^{\infty} u(x)\,dx$ converge or diverge together.

Let us now test the convergence of the infinite integral.

$\because$
$$\int_1^x u(x)dx = \int_1^x \frac{1}{x^p}dx = \begin{cases} \dfrac{1}{1-p}(x^{1-p} - 1), & \text{if } p \neq 1 \\ \log x, & \text{if } p = 1 \end{cases}$$

$\therefore$
$$\int_1^{\infty} u(x)dx = \lim_{x\to\infty} \int_1^x u(x)\,dx = \begin{cases} \dfrac{1}{p-1}, & \text{if } p > 1 \\ \infty, & \text{if } 0 < p \leq 1 \end{cases}$$

Thus $\int_1^\infty u(x)\,dx$ converges, if $p > 1$, and diverges, if $0 < p \le 1$.

Hence, the infinite series $\sum(1/n^p)$ converges if $p > 1$, and diverges if $0 < p \le 1$.

But, when $p < 0$, the series $\sum(1/n^p)$ diverges for then the nth term n^{-p} does not tend to zero as $n \to \infty$.

Hence the series $\sum(1/n^p)$ converges when $p > 1$, and diverges when $p \le 1$.

Example 4.15. The series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$, $p > 0$, converges for $p > 1$ and diverges for $p \le 1$.

Solution. Let $u(x) = \frac{1}{x(\log x)^p}$, so that for $x \ge 2$, the function u is a non-negative monotonic decreasing integrable function, such that

$$u_n = u(n) = \frac{1}{n(\log n)^p}, \quad \forall\ p > 0, \quad n \in \mathbf{N}$$

By Integral Test $\sum_{n=2}^{\infty} u_n$ and $\int_2^\infty u(x)\,dx$ converge or diverge together.

Let us now test the convergence of the infinite integral.

$$\because \qquad \int_2^X u(x)\,dx = \int_2^X \frac{1}{x(\log x)^p}\,dx, \quad p > 0$$

$$= \begin{cases} \dfrac{(\log X)^{1-p} - (\log 2)^{1-p}}{1-p}, & \text{if } p \ne 1, \\ \log\log X - \log\log 2, & \text{if } p = 1. \end{cases}$$

$$\therefore \qquad \int_2^\infty u(x)\,dx = \lim_{X \to \infty} \int_2^X u(x)\,dx = \begin{cases} \dfrac{(log\ 2)^{1-p}}{p-1}, & \text{if } p > 1, \\ \infty, & \text{if } 0 < p \le 1. \end{cases}$$

Thus $\int_2^\infty u(x)\,dx$ converges if $p > 1$, and diverges if $0 < p \le 1$.

Hence, the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$, $p > 0$, converges if $p > 1$, and diverges if $p \le 1$.

4.9 GAUSS'S TEST

If $\sum u_n$ *is a positive term series such that,*

$$\frac{u_n}{u_{n+1}} = \alpha + \frac{\beta}{n} + \frac{\gamma_n}{n^p},$$

where $\alpha > 0, p > 1,$ *and* $\{\gamma_n\}$ *is a bounded sequence, then*

(*i*) *for* $\alpha \neq 1, \Sigma u_n$ *converges if* $\alpha > 1,$ *and diverges if* $\alpha < 1,$ *whatever* β *may be,*

(*ii*) *for* $\alpha = 1, \Sigma u_n$ *converges if* $\beta > 1$ *and diverges if* $\beta \leq 1.$

Proof: (*i*) When $\alpha \neq 1,$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \alpha$$

Hence by Ratio Test, the series converges if $\alpha > 1,$ and diverges if $\alpha < 1.$

(*ii*) When $\alpha = 1,$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \beta$$

Hence by Raabe's Test, the series converges if $\beta > 1,$ and diverges if $\beta < 1.$

For $\beta = 1,$ we have

$$\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{\gamma_n}{n^p}, \quad p > 1$$

Let us compare the given series with the divergent series $\sum v_n$ where $v_n = \frac{1}{n \log n}.$

Now

$$\frac{u_n}{u_{n+1}} - \frac{v_n}{v_{n+1}} = 1 + \frac{1}{n} + \frac{\gamma_n}{n^p} - \frac{(n+1)\log(n+1)}{n \log n}$$

$$= \frac{\gamma_n}{n^p} - \frac{n+1}{n}\left[\frac{\log(n+1)}{\log n} - 1\right]$$

$$= \frac{1}{n^p}\left[\gamma_n - (n+1)\log\left(1 + \frac{1}{n}\right) \cdot \frac{n^{p-1}}{\log n}\right]$$

Since $$\lim_{n=\infty} (n+1)\log\left(1 + \frac{1}{n}\right) = \lim_{n\to\infty}\left[\log\left(1 + \frac{1}{n}\right)^n + \log\left(1 + \frac{1}{n}\right)\right] = 1$$

and $\lim_{n=\infty} \frac{n^{p-1}}{\log n} = \infty, p > 1,$ and $\{\gamma_n\}$ is bounded, therefore, for sufficiently large values of $n,$

$\gamma_n - (n+1)\log\left(1 + \frac{1}{n}\right) \cdot \frac{n^{p-1}}{\log n}$ remains negative.

Thus $\exists$ a positive integer m such that

$$\gamma_n - (n+1)\log\left(1+\frac{1}{n}\right)\cdot\frac{n^{p-1}}{\log n} < 0, \qquad \forall\ n \geq m$$

$$\Rightarrow \qquad \frac{u_n}{u_{n+1}} - \frac{v_n}{v_{n+1}} < 0, \qquad \forall\ n \geq m$$

$$\Rightarrow \qquad \frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}, \qquad \forall\ n \geq m$$

Since $\sum v_n$ is divergent, therefore by Comparison Test, the series $\sum u_n$ is also divergent.

Remarks 1. Gauss's Test is very useful and may be used after the failure of Raabe's Test or directly without recourse to other tests.

2. If

$$\frac{u_n}{u_{n+1}} = \alpha + \frac{\beta}{n} + \frac{\gamma}{n^2} + \frac{\delta}{n^3} + \ldots \qquad \ldots(1)$$

where $\alpha, \beta, \gamma \ldots$ are independent of n, then we can write

$$\frac{u_n}{u_{n+1}} = \alpha + \frac{\beta}{n} + \frac{\gamma_n}{n^2}$$

where $\gamma_n = \gamma + \delta/n + \ldots$, so that $\lim \gamma_n = \gamma$, *i.e.*, $\{\gamma_n\}$ is a bounded sequence.

Thus for the application of Gauss's Test we may expand u_n/u_{n+1} in powers of $1/n$ as in (1).

Example 4.16. Test the convergence of the series

$$\frac{2^2}{3^2} + \frac{2^2\cdot 4^2}{3^2\cdot 5^2} + \frac{2^2\cdot 4^2\cdot 6^2}{3^2\cdot 5^2\cdot 7^2} + \ldots$$

Solution. Now

$$u_n = \frac{2^2\cdot 4^2\cdot 6^2 \ldots (2n)^2}{3^2\cdot 5^2\cdot 7^2 \ldots (2n+1)^2}$$

$$\therefore \qquad \frac{u_n}{u_{n+1}} = \frac{(2n+3)^2}{(2n+2)^2} = \left(1+\frac{3}{2n}\right)^2\left(1+\frac{1}{n}\right)^{-2}$$

$$= \left(1+\frac{3}{n}+\frac{9}{4n^2}\right)\left(1-\frac{2}{n}+\frac{3}{n^2}-\ldots\right)$$

$$= 1 + \frac{1}{n} - \frac{3}{4n^2} + \ldots \text{ higher powers of } \frac{1}{n},$$

so that $\alpha = 1$ and $\beta = 1$.

Hence by Gauss's Test, the series diverges.

Example 4.17. Test for convergence of the series

$$\Sigma \frac{1^2 \cdot 3^2 \ldots (2n-1)^2}{2^2 \cdot 4^2 \ldots (2n)^2} x^{n-1}, \quad x > 0.$$

Solution. Here

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(2n+2)^2}{(2n+1)^2} \cdot \frac{1}{x} = \frac{1}{x}$$

Hence by Ratio Test, the series converges if $x < 1$, and diverges if $x > 1$

Now for $x = 1$,

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2}$$

$$\therefore \qquad \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{4n^2 + 3n}{(2n+1)^2} = 1$$

Hence Raabe's Test fails.

Let us now apply Gauss's Test.

$$\frac{u_n}{u_{n+1}} = \left(1 + \frac{1}{n}\right)^2 \left(1 + \frac{1}{2n}\right)^{-2}$$

$$= \left(1 + \frac{2}{n} + \frac{1}{n^2}\right)\left(1 - \frac{1}{n} + \frac{3}{4n^2} + \ldots\right)$$

$$= 1 + \frac{1}{n} - \frac{1}{4n^2} + \ldots \text{ higher powers of } \frac{1}{n}$$

so that by Gauss's Test, the series diverges.

Hence, the series converges for $x < 1$, and diverges for $x \geq 1$.

Note. We could get the result directly by Gauss's Test, for

$$\frac{u_n}{u_{n+1}} = \frac{1}{x}\left(\frac{2n+2}{2n+1}\right)^2 = \frac{1}{x} + \frac{1/x}{n} - \frac{1/4x}{n^2} + \ldots$$

where $\alpha = 1/x$, $\beta = 1/x$.

Example 4.18. Test for convergence of the hypergeometric series

$$1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1)\,\beta(\beta+1)}{1 \cdot 2\gamma(\gamma+1)} x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\,\beta(\beta+1)(\beta+2)}{1 \cdot 2 \cdot 3\,\gamma(\gamma+1)(\gamma+2)} x^3 + \ldots$$

for all positive values of x; α, β, γ being all positive.

Solution. It is a positive term series.

Ignoring the first term, which does not affect the behaviour of the series, we have

$$u_n = \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{1\cdot 2\cdot 3\ldots n}\,\frac{\beta(\beta+1)\ldots(\beta+n-1)}{\gamma(\gamma+1)\ldots(\gamma+n-1)}\,x^n$$

so that

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)}\,\frac{1}{x} = \frac{1}{x}$$

Hence, by Ratio Test, the series converges, if $x < 1$, and diverges, if $x > 1$.

For $x = 1$, we have

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)} = \frac{\left(1+\frac{\gamma+1}{n}+\frac{\gamma}{n^2}\right)}{\left(1+\frac{\alpha+\beta}{n}+\frac{\alpha\beta}{n^2}\right)}$$

$$= \left(1+\frac{\gamma+1}{n}+\frac{\gamma}{n^2}\right)\left[1-\left(\frac{\alpha+\beta}{n}+\frac{\alpha\beta}{n^2}\right)+\left(\frac{\alpha+\beta}{n}+\frac{\alpha\beta}{n^2}\right)^2+\ldots\right]$$

$$= 1+\frac{1+\gamma-\alpha-\beta}{n}+\frac{(\alpha+\beta-\gamma)(\alpha+\beta-1)-\alpha\beta}{n^2}+\ldots$$

Hence by Gauss's Test the series converges if $1+\gamma-\alpha-\beta>1$ or $\gamma>\alpha+\beta$ and diverges if $1+\gamma-\alpha-\beta\le 1$ or $\gamma\le\alpha+\beta$.

Thus for positive values of α, β, γ and x,

(*i*) for $x < 1$, the series converges,

(*ii*) for $x > 1$, the series diverges, and

(*iii*) for $x = 1$, the series converges if $\gamma > (\alpha+\beta)$ and diverges if $\gamma \le (\alpha+\beta)$.

EXERCISE

Test the convergence of the series:

1. $1+\frac{2x}{2!}+\frac{3^2\cdot x^2}{3!}+\frac{4^3\cdot x^3}{4!}+\ldots$

2. $\frac{x}{1}+\frac{1}{2}\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7}+\ldots$

3. $\frac{a+x}{1!}+\frac{(a+2x)^2}{2!}+\frac{(a+3x)^3}{3!}+\ldots,$ for $x>0$.

4. $\frac{1^2}{2^2}+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}+\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}+\ldots$

5. Apply Cauchy's Integral Test to test the convergence of the series.

   (*i*) $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$, (*ii*) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

6. Test the convergence of $\sum_{n=3}^{\infty} \frac{1}{n \log n \left(\log \log n\right)^p}$

Test the convergence of the series whose *n*th term is

7. $\frac{(n!)}{(2n)!} x^n$

8. $\frac{n!}{(n+1)^n} x^n$

9. Prove that $1 + \frac{\alpha}{\beta} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)} + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)} + \ldots$, where x, are positive, converges if $\beta > \alpha + 1$, and diverges if $\beta \leq \alpha + 1$.

ANSWERS

1. Converges if $x < 1/e$, diverges if $x \geq 1/e$.
2. Converges if $|x| \leq 1$, diverges if $|x| > 1$.
3. Converges for $x < 1/e$, diverges for $x \geq 1/e$.
4. Diverges
5. (*i*) Convergent (*ii*) Convergent.
6. Converges for $p > 1$, diverges for $p \leq 1$.
7. Converges for $x < 4$, diverges for $x \geq 4$.
8. Converges for $0 < x < e$, diverges for $x \geq e$.

4.10 SERIES WITH ARBITRARY TERMS

So far we have considered series with positive terms only. We shall now discuss series with terms having any sign whatsoever.

4.10.1 Alternating Series

A series whose terms are alternatively positive and negative, *e.g.*,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

is called an *alternating series*.

Leibnitz Test. *If the alternating series*

$$u_1 - u_2 + u_3 - u_4 + \ldots, \left(u_n > 0, \forall\ n\right)$$

is such that

(*i*) $u_{n+1} \leq u_n, \ \forall\ n$, *and*

(*ii*) $\lim_{n\to\infty} u_n = 0$, *then the series converges.*

Let $S_n = u_1 - u_2 + u_3 - u_4 + ... + (-1)^{n-1}\, u_n$

Now for all n,

$$S_{2n+2} - S_{2n} = u_{2n+1} - u_{2n+2} \geq 0$$

$$\Rightarrow \qquad S_{2n+2} \geq S_{2n}$$

$\Rightarrow \{S_{2n}\}$ is a monotonic increasing sequence.

Again

$$\begin{aligned} S_{2n} &= u_1 - u_2 + u_3 - ... + u_{2n-1} - u_{2n} \\ &= u_1 - (u_2 - u_3) - (u_4 - u_5) - ... - (u_{2n-2} - u_{2n-1}) - u_{2n} \end{aligned}$$

But since $u_{n+1} \leq u_n$, for all n, therefore, each bracket on the right is positive and hence

$$S_{2n} < u_1, \ \forall\ n$$

Thus, the monotonic increasing sequence $\{S_{2n}\}$ is bounded above and is consequently convergent.

Let $\lim_{n\to\infty} S_{2n} = S$.

We shall now show that the sequence $\{S_{2n+1}\}$ also converges to the same limit S.

Now

$$S_{2n+1} = S_{2n} + u_{2n+1}$$

$$\therefore \qquad \lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} u_{2n+1}$$

But by condition (*ii*),

$$\lim_{n\to\infty} u_{2n+1} = 0$$

$$\therefore \qquad \lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} = S$$

Thus, the sequences $\{S_{2n}\}$ and $\{S_{2n+1}\}$ both converge to the same limit S. We shall now show that the sequence $\{S_n\}$ also converges to S.

Let $\varepsilon > 0$ be given.

Since the sequences $\{S_{2n}\}$ and $\{S_{2n+1}\}$ both converge to S, therefore $\exists$ positive integers m_1, m_2, respectively, such that

$$\left| S_{2n} - S \right| < \varepsilon, \qquad \forall\ n \geq m_1 \qquad ...(1)$$

and

$$\left| S_{2n+1} - S \right| < \varepsilon, \qquad \forall\ n \geq m_2 \qquad ...(2)$$

Thus from (1) and (2) we have

$$|S_n - S| < \varepsilon, \quad \forall\ n \geq \max(2m_1, 2m_2 + 1)$$

$\Rightarrow$ $\{S_n\}$ converges to S

$\Rightarrow$ the series $\sum(-1)^{n-1}\ u_n$ converges.

Example 4.19. Show that the series $\frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \ldots$ converges for $p > 0$.

Solution. Let $u_n = 1/n^p$.

Here

$$u_{n+1} \leq u_n; \quad \forall\ n$$

and

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n^p} = 0$$

Hence by Leibnitz test, the alternating series $\sum \frac{(-1)^{n-1}}{n^p}$ converges.

4.10.2 Absolute Convergence

A series $\sum u_n$ is said to be *absolutely convergent* if the series $\sum u_n$ obtained on taking every term of the given series with a positive sign is convergent, *i.e.*, if the series $\sum |u_n|$ is convergent.

Conditional Convergence. A series which is convergent but is not absolutely convergent is called a *conditionally convergent* series.

ILLUSTRATIONS

1. The series

$$1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \ldots,$$

is absolutely convergent because the series

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \ldots,$$

obtained on taking every term of the given series with a positive sign, is convergent.

2. The series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

is convergent by Leibnitz test, but the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

obtained on taking every term with a positive sign is divergent. Thus the given series is conditionally convergent.

Theorem 4.9. *Every absolutely convergent series is convergent.*

Let $\sum u_n$ be absolutely convergent, so that $\sum |u_n|$ is convergent.

$\therefore$ For any $\varepsilon > 0$, by Cauchy's General Principle of convergence, $\exists$ a positive integer m such that

$$|u_{n+1}| + |u_{n+2}| + ... + |u_{n+p}| < \varepsilon, \quad \forall\ n \geq m \wedge p \geq 1$$

Also for all n and $p > 1$,

$$|u_{n+1} + u_{n+2} + ... + u_{n+p}| \leq |u_{n+1}| + |u_{n+2}| + ... + |u_{n+p}| < \varepsilon, \quad \forall\ n \geq m \wedge p \geq 1.$$

Hence, by Cauchy's General Principle of convergence the series $\sum u_n$ converges.

Aliter. Let $\sum u_n$ be absolutely convergent, so that $\sum |u_n|$ is convergent.

Let $a_n = \begin{cases} u_n, & \text{if } u_n \geq 0 \\ 0, & \text{if } u_n < 0 \end{cases}$ and $b_n = \begin{cases} -u_n, & \text{if } u_n < 0 \\ 0, & \text{if } u_n \geq 0. \end{cases}$

Then clearly,

$$a_n \geq 0, b_n \geq 0, \qquad ...(i)$$

$$u_n = a_n - b_n \qquad ...(ii)$$

and

$$|u_n| = a_n + b_n \qquad ...(iii)$$

From (*i*) and (*iii*) it follows that

$$a_n \leq |u_n|, b_n \leq |u_n|$$

Since $\sum |u_n|$ is convergent, therefore, by Comparison Test, both $\sum a_n$ and $\sum b_n$ are convergent.

Hence, by Theorem 4.5, $\sum (a_n - b_n)$ is convergent.

$\therefore$ From (*ii*), it follows that $\sum u_n$ is convergent.

Remarks 1. The divergence of $\sum |u_n|$ does not imply the divergence of $\sum u_n$.

For example, if $u_n = \dfrac{(-1)^{n-1}}{n}$, we have seen above that $\sum |u_n|$ is divergent, whereas $\sum u_n$ is convergent.

2. The very great significance of the concept of Absolute Convergence is that the convergence of absolutely convergent series is much more easy to recognise than that of conditionally convergent series—usually by comparison with series of positive terms. In fact all the tests for positive term series become available for the purpose. But this significance becomes all the more visible in the discussion of rearrangement of series—so much so that we may operate on absolutely convergent series, precisely as we operate on sums of a finite number of terms, whereas in the case of conditionally convergent series this in general is not possible.

Example 4.20. Show that for any fixed value of x, the series $\sum_{n=1}^{\infty} \dfrac{\sin nx}{n^2}$ is convergent.

Solution. Let $u_n = \dfrac{\sin nx}{n^2}$, so that $|u_n| = \dfrac{|\sin nx|}{n^2}$

Now $\left|\dfrac{\sin nx}{n^2}\right| \le \dfrac{1}{n^2}, \forall\ n$ and $\Sigma\dfrac{1}{n^2}$ converges.

Hence, by Comparison Test, the series $\Sigma\left|\dfrac{\sin nx}{n^2}\right|$ converges.

Since every absolutely convergent series is convergent, therefore $\Sigma\dfrac{\sin nx}{n^2}$ is convergent.

Example 4.21. Show that the series

$$x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

converges absolutely for all values of x.

Solution. Let $u_n = \dfrac{x^n}{n!}$.

Now $\lim\limits_{n\to\infty} \dfrac{|u_n|}{|u_{n+1}|} = \lim\limits_{n\to\infty} \dfrac{n+1}{|x|} \to \infty$ except when $x = 0$.

So by Ratio Test, the series converges absolutely for all x except possibly zero.

But for $x = 0$ the series evidently converges absolutely.

Hence, the series converges absolutely for all values of x.

Note. Since for a convergent series Σu_n, $\lim\limits_{n\to\infty} u_n = 0$,

$\therefore$ $\lim\limits_{n\to\infty} \dfrac{x^n}{n!} = 0$ a useful result.

Example 4.22. Show that

$$\lim_{n\to\infty} \frac{m(m-1)\ldots(m-n+1)}{(n-1)!}\, x^n = 0,$$

where $|x| < 1$ and m is any real number.

Solution. Consider the series Σu_n, where

$$u_n = \frac{m(m-1)\ldots(m-n+1)}{(n-1)!}\, x^n$$

Now

$$\lim_{n\to\infty}\frac{|u_n|}{|u_{n+1}|}=\lim_{n\to\infty}\left|\frac{n}{m-n}\right|\cdot\frac{1}{|x|}$$

$$=\lim_{n\to\infty}\left|\frac{1}{\frac{m}{n}-1}\right|\cdot\frac{1}{|x|}=\frac{1}{|x|}$$

Hence, the series $\sum u_n$ converges absolutely for $|x|<1$

$\Rightarrow$ the series $\sum u_n$ converges for $|x|<1$

$\Rightarrow$ $\lim_{n\to\infty} u_n=0,$ for $|x|<1$

i.e., $$\lim_{n\to\infty}\frac{m(m-1)\ldots(m-n+1)}{(n-1)!}x^n=0,\ \text{if}\ |x|<1$$

Note. The results of Example 4.21 and 4.22 are very useful.

Also see Examples 3.9 and 3.10 of Ch. 3.

EXERCISE

1. Show that the following series are convergent:

   (*i*) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$

   (*ii*) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$

   (*iii*) $1-\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}-\frac{1}{4\sqrt{4}}+\ldots$

   (*iv*) $\frac{\log 2}{2^2}-\frac{\log 3}{3^2}+\frac{\log 4}{4^2}-\ldots$

2. Prove that the following series are absolutely convergent:

   (*i*) $1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots$

   (*ii*) $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$

   (*iii*) $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots$

3. Show that $\lim_{n\to\infty}\frac{n^r}{x^n}=0$

4. Show that the following series are conditionally convergent:

   (*i*) $\Sigma \dfrac{(-1)^{n+1}}{\sqrt{n}}$

   (*ii*) $\Sigma \dfrac{(-1)^{n+1}}{3n-2}$

5. Show that the series $\Sigma \dfrac{(-1)^{n+1}}{n^p}$ is absolutely convergent for $p > 1$, but conditionally convergent for $0 < p \leq 1$.

6. Show that the following series are absolutely convergent:

   (*i*) $\Sigma (-1)^{n-1} \left\{ \dfrac{1}{n^2} + \dfrac{1}{(n+1)^2} \right\}$

   (*ii*) $\Sigma (-1)^{n-1} \left\{ \dfrac{1}{n^{5/2}} + \dfrac{1}{(n+1)^{5/2}} \right\}$

   (*iii*) $\Sigma (-1)^n \dfrac{n+2}{2^n+5}$

7. Show that the series $\Sigma \left(\dfrac{1}{n} + \dfrac{(-1)^{n-1}}{\sqrt{n}} \right)$ is divergent.

8. Show that the series $1 - \dfrac{1}{3 \cdot 4} + \dfrac{1}{5 \cdot 4^2} - \dfrac{1}{7 \cdot 4^3} + \ldots$ converges.

CHAPTER 5

Functions with Interval as Domain (I)

LIMIT AND CONTINUITY

In the chapter on sequences, we considered functions whose domain was the set **N** of natural numbers. We shall now consider real valued functions with domain as any interval, open or closed.

5.1 LIMITS

Let f be a function defined for all points in a neighbourhood N of a point c except possibly at the point c itself.

Definition 1. The function f is said to tend to a limit l as x tends to (or approaches) c if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x - c| < \delta \Rightarrow |f(x) - l| < \varepsilon$$

or $$|f(x) - l| < \varepsilon, \text{ when } 0 < |x - c| < \delta$$

or $$f(x) \in]l - \varepsilon, l + \varepsilon[, \forall\, x \in]c - \delta, c + \delta[\text{ except possibly } c$$

In symbols, we then write

$$\lim_{x \to c} f(x) = l$$

Definition 2. The function f is said to tend to $+\infty$ as x tends to c (or in symbols, $\lim_{x \to c} f(x) = +\infty$) if for each $G > 0$ (however large) there exists a $\delta > 0$ such that

$$f(x) > G, \text{ when } |x - c| < \delta$$

The function f is said to tend to $-\infty$ as x tends to c (or in symbols, $\lim_{x \to c} f(x) = -\infty$) if for each $G > 0$ (however large) there exists a $\delta > 0$ such that

$$f(x) < -G, \text{ when } |x - c| < \delta$$

Definition 3. The function f is said to tend to a limit l as x tends to ∞ (or in symbols, $\lim_{x \to \infty} f(x) = l$) if for each $\varepsilon > 0$, there exists a $k > 0$,

such that $$|f(x) - l| < \varepsilon, \text{ when } x > k$$

Definition 4. The function f is said to tend to $+\infty$ as x tends to ∞ (or in symbols, $\lim_{x\to\infty} f(x) = \infty$) if for each $G > 0$ (however large) there exists a $k > 0$, such that

$$f(x) > G, \text{ when } x > k$$

5.1.1 Left Hand and Right Hand Limits

While defining the limit of a function f as x tends to c, we consider values of $f(x)$ when x is very close to c. The values of x may be greater or less than c. If we restrict x to values less than c, then we say that x tends to c from below or from the left and write it symbolically as $x \to c - 0$ or simply $x \to c-$. The limit of f with this restriction on x, is called the *left hand limit*. Similarly, if x takes only values greater than c, then x is said to tend to c from above or from the right, and is denoted symbolically as $x \to c + 0$ or $x \to c +$. The limit of f is then called the *right hand limit*.

Definition 5. A function f is said to tend to a limit l as x tends to c from the left if for each $\varepsilon > 0$, there exists $\delta > 0$, such that

$$|f(x) - l| < \varepsilon, \text{ when } c - \delta < x < c.$$

In symbols, we then write

$$\lim_{x\to c-0} f(x) = l \text{ or } f(c - 0) = l \text{ or } f(c-) = l.$$

Definition 6. A function f is said to tend to a limit l as x tends to c from the right if for each $\varepsilon > 0$, there exists $\delta > 0$, such that

$$|f(x) - l| < \varepsilon, \text{ when } c < x < c + \delta.$$

In symbols, we then write

$$\lim_{x\to c+0} f(x) = l \text{ or } f(c + 0) = l \text{ or } f(c+) = l.$$

It may be noted that $\lim_{x\to c} f(x)$ exists if and only if both the limits, the left hand and the right hand, exist and are equal.

One-sided infinite limit may also be defined in the same way as above.

Example 5.1. Find the right hand and the left hand limits of a function defined as follows:

$$f(x) = \begin{cases} \dfrac{|x-4|}{x-4}, & x \neq 4, \\ 0, & x = 4. \end{cases}$$

Solution. Now, when $x > 4$, $|x - 4| = x - 4$.

$$\therefore \quad \lim_{x\to 4+0} f(x) = \lim_{x\to 4+0} \frac{|x-4|}{x-4} = \lim_{x\to 4+0} \frac{x-4}{x-4} = \lim_{x\to 4+0} 1 = 1$$

Again when $x < 4$, $|x - 4| = -(x - 4)$.

$$\therefore \quad \lim_{x\to 4-0} f(x) = \lim_{x\to 4-0} \frac{-(x-4)}{x-4} = \lim_{x\to 4-0} (-1) = -1.$$

So that

$$\lim_{x\to 4-0} f(x) \neq \lim_{x\to 4+0} f(x)$$

$\therefore \lim_{x\to 4+} f(x)$ does not exist.

Example 5.2. Evaluate $\lim_{x\to 0+} \frac{1}{1+e^{-1/x}}$.

Solution. [As $x \to 0+$, we feel that $1/x$ increases indefinitely, $e^{1/x}$ increases indefinitely. $e^{-1/x}$ tends to 0, $1 + e^{-1/x}$ tends to 1; thus the required limit may be 1.]

We have to show that for a given $\varepsilon > 0, \exists$ a $\delta > 0$, such that

$$\left|\frac{1}{1+e^{-1/x}} - 1\right| < \varepsilon, \text{ when } 0 < x < \delta$$

Now

$$\left|\frac{1}{1+e^{-1/x}} - 1\right| = \left|\frac{-e^{-1/x}}{1+e^{-1/x}}\right| = \frac{1}{e^{1/x}+1} < \varepsilon,$$

when $e^{1/x} + 1 > \frac{1}{\varepsilon}$ or $\frac{1}{x} > \log\left(\frac{1}{\varepsilon} - 1\right)$

$$\Rightarrow \qquad 0 < x < \frac{1}{\log(1/\varepsilon - 1)}, \text{ for } 0 < \varepsilon < 1.$$

Thus choosing $\delta = \frac{1}{\log(1/\varepsilon - 1)}$, we see that if $0 < \varepsilon < 1$,

$$\left|\frac{1}{1+e^{-1/x}} - 1\right| < \varepsilon, \text{ when } 0 < x < \delta$$

Again when $\varepsilon \geq 1$,

$$\left|\frac{1}{1+e^{-1/x}} - 1\right| < \varepsilon \Rightarrow e^{1/x} > \frac{1}{\varepsilon} - 1.$$

which is true for all values of x, so that any $\delta > 0$ would work.

Thus for any $\varepsilon > 0$ we are able to find a $\delta > 0$, such that

$$\left|\frac{1}{1+e^{-1/x}} - 1\right| < \varepsilon, \text{ when } 0 < x < \delta.$$

$$\therefore \qquad \lim_{x\to 0+} \frac{1}{1+e^{-1/x}} = 1.$$

Example 5.3. Prove that $\lim_{x\to 0} x\sin\frac{1}{x} = 0$.

Solution. Now

$$\left| x\sin\frac{1}{x} \right| = |x| \cdot \left| \sin\frac{1}{x} \right| \le |x|$$

Thus choosing a $\delta = \varepsilon$, we see that

$$\left| x\ \sin\frac{1}{x} \right| < \varepsilon, \text{ when } 0 < |x| < \delta$$

$$\Rightarrow \qquad \lim_{x\to 0} x\sin\frac{1}{x} = 0$$

Example 5.4. If $\lim_{x\to a} f(x)$ exists, prove that it must be unique.

Solution. Let, if possible, $f(x)$ tends to limits l_1 and l_2.

Hence for any $\varepsilon > 0$, it is possible to choose a $\delta > 0$, such that

$$|f(x) - l_1| < \frac{1}{2}\varepsilon, \text{ when } 0 < |x - a| < \delta$$

$$|f(x) - l_2| < \frac{1}{2}\varepsilon, \text{ when } 0 < |x - a| < \delta$$

Now

$$\begin{aligned} |l_1 - l_2| &= |l_1 - f(x) + f(x) - l_2| \\ &\le |l_1 - f(x)| + |f(x) - l_2| < \varepsilon, \end{aligned}$$

when $\qquad 0 < |x - a| < \delta$

i.e., $|l_1 - l_2|$ is less than any positive number (however small) and so must be equal to zero. Thus, $l_1 = l_2$.

Example 5.5. Show that $\lim_{x\to 3} \frac{1}{(x-3)^4} = \infty$.

Solution. Let G be any positive number, however large.

Now $\left| \frac{1}{(x-3)^4} \right| > G$

or

$$\frac{1}{(x-3)^4} > G, \text{ when } (x-3)^4 < \frac{1}{G} \text{ or when } 0 < |x-3| < \frac{1}{G^{1/4}}$$

Choosing $\delta = \frac{1}{G^{1/4}}$, we get the required result.

5.1.2 Theorems on Limits

Let f and g be two real functions with domain D. We define four new functions $f \pm g$, fg, f/g on domain D by setting

$$(f+g)\,x = f(x) + g(x)$$
$$(f-g)\,x = f(x) - g(x)$$
$$(fg)\,x = f(x) \cdot g(x)$$
$$(f/g)\,x = f(x)/g(x), \text{ if } g(x) \neq 0 \text{ for any } x \in D$$

We shall now prove a theorem concerning the limits of these functions.

Theorem 5.1. *If f and g be two functions defined on some neighbourhood of c such that*

$$\lim_{x\to c} f(x) = l, \lim_{x\to c} g(x) = m,$$

then

(*i*) $\lim_{x\to c}(f+g)x = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = l + m.$

(*ii*) $\lim_{x\to c}(f-g)x = \lim_{x\to c} f(x) - \lim_{x\to c} g(x) = l - m.$

(*iii*) $\lim_{x\to c}(fg)x = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = lm.$

(*iv*) $\lim_{x\to c}(f/g)x = \lim_{x\to c} f(x) / \lim_{x\to c} g(x) = l/m, \text{ if } m \neq 0.$

(*i*) Since $\lim_{x\to c} f(x) = l$, $\lim_{x\to c} g(x) = m$, therefore for any $\varepsilon > 0$, $\exists$ positive numbers δ_1, δ_2, such that

$$|f(x) - l| < \frac{1}{2}\varepsilon, \quad \text{when } 0 < |x-c| < \delta_1$$

$$|g(x) - m| < \frac{1}{2}\varepsilon, \quad \text{when } 0 < |x-c| < \delta_2.$$

If $\delta = min(\delta_1, \delta_2)$, then for $0 < |x-c| < \delta$,

$$|f(x) - l| < \frac{1}{2}\varepsilon, |g(x) - m| < \frac{1}{2}\varepsilon$$

and

$$|(f+g)x - (l+m)| = |f(x) - l + g(x) - m|$$

$$\leq |f(x) - l| + |g(x) - m|$$
$$< \varepsilon$$

$\Rightarrow$ $|(f + g)x - (l + m)| < \varepsilon$, when $0 < |x - c| < \delta$

$\Rightarrow$ $\lim_{x \to c} (f + g)x = l + m$

(*ii*) Proof is similar to part (*i*).

(*iii*) Let $\varepsilon > 0$ be given.

Now

$$|(fg)x - lm| = |f(x)(g(x) - m) + m(f(x) - l)|$$
$$\leq |f(x)| \cdot |g(x) - m| + |m| \cdot |f(x) - l| \quad \ldots(1)$$

Since $\lim_{x \to c} f(x) = l$, therefore, for $\varepsilon = 1$, $\exists$ *a* $\delta_1 > 0$, such that

$$|f(x) - l| < 1, \text{ when } 0 < |x - c| < \delta_1.$$

Now

$$|f(x)| = |f(x) - l + l| \leq |f(x) - l| + |l|$$
$$< 1 + |l|, \text{ when } 0 < |x - c| < \delta_1 \quad \ldots(2)$$

Again, since $\lim_{x \to c} g(x) = m$, therefore, $\exists$ a $\delta_2 > 0$, such that

$$|g(x) - m| < \frac{\frac{1}{2}\varepsilon}{1 + |l|}, \text{ when } 0 < |x - c| < \delta_2 . \quad \ldots(3)$$

Also, as $\lim_{x \to c} f(x) = l$, therefore, $\exists$ a $\delta_3 > 0$, such that

$$|f(x) - l| < \frac{\frac{1}{2}\varepsilon}{1 + |m|}, \text{ when } 0 < |x - c| < \delta_3. \quad \ldots(4)$$

Let $\delta = \min(\delta_1, \delta_2, \delta_3)$.

Then, from (1), (2), (3) and (4), we have

$$|(fg)x - lm| < \frac{(1 + |l|)\frac{1}{2}\varepsilon}{1 + |l|} + \frac{|m|\varepsilon/2}{1 + |m|} < \varepsilon,$$

when $0 < |x - c| < \delta$.

Hence,

$$\lim_{x \to c} (fg)(x) = lm$$

(*iv*) *Lemma.* Show that if $\lim_{x \to c} g(x) = m > 0$, then $\exists$ a $\delta_1 > 0$ such that

$$|g(x)| > \frac{1}{2}|m|, \text{ when } 0 < |x - c| < \delta_1$$

Since $\lim_{x \to c} g(x) = m$, therefore, for $\varepsilon = \frac{1}{2}|m| > 0$, $\exists$ a $\delta_1 > 0$, such that

$$|g(x) - m| < \frac{1}{2}|m|, \text{ when } 0 < |x - c| < \delta_1$$

Also

$$|m| = |m - g(x) + g(x)| \leq |m - g(x)| + |g(x)|$$

$$< \frac{1}{2}|m| + |g(x)|, \text{ when } 0 < |x - c| < \delta_1$$

i.e., $$|g(x)| > \frac{1}{2}|m|, \text{ when } 0 < |x - c| < \delta_1$$

$\Rightarrow \exists$ a deleted neighbourhood of c on which $g(x)$ does not vanish.

Let us now attend to the main theorem.

$$|(f/g)x - l/m| = \frac{mf(x) - lg(x)}{mg(x)}$$

$$\leq \frac{|f(x) - l|}{|g(x)|} + \frac{|l| \cdot |g(x) - m|}{|m| \cdot |g(x)|}$$

$$< \frac{2|f(x) - l|}{|m|} + \frac{2|l| \cdot |g(x) - m|}{|m|^2} \qquad ...(1)$$

when $0 < |x - c| < \delta_1$

Let $\varepsilon > 0$ be given.

Since $\lim_{x \to c} f(x) = l$, $\lim_{x \to c} g(x) = m$, therefore, $\exists$ positive numbers δ_2 and δ_3, such that

$$|f(x) - l| < \frac{1}{4}\varepsilon|m|, \text{ when } 0 < |x - c| < \delta_2 \qquad ...(2)$$

and

$$|g(x) - m| < \frac{1}{4}\varepsilon \frac{|m|^2}{|l| + 1}, \text{ when } 0 < |x - c| < \delta_3 \qquad ...(3)$$

Let $\delta = \min(\delta_1, \delta_2, \delta_3)$.

Thus, from (1), (2) and (3), we have

$$|(f/g)x - l/m| < \varepsilon, \text{ when } 0 < |x - c| < \delta$$

Hence, $\lim_{x \to c} (f/g)x = l/m$, provided $m \neq 0$.

Example 5.6. Evaluate:

$$(i)\ \lim_{x\to -1}\frac{(x+2)(3x-1)}{x^2+3x-2},$$

$$(ii)\ \lim_{x\to 0}\frac{\sqrt{4+x}-2}{x},$$

$$(iii)\ \lim_{x\to 0+}\frac{\sin x}{\sqrt{x}}.$$

Solution.

$$(i)\ \lim_{x\to -1}\frac{(x+2)(3x-1)}{x^2+3x-2}=\frac{\lim\limits_{x\to -1}(x+2)\cdot\lim\limits_{x\to -1}(3x-1)}{\lim\limits_{x\to -1}\left(x^2+3x-2\right)}$$

$$=\frac{1\cdot(-4)}{-4}=1.$$

$$(ii)\ \lim_{x\to 0}\frac{\sqrt{4+x}-2}{x}=\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x}\cdot\frac{\sqrt{4+x}+2}{\sqrt{4+x}+2}$$

$$=\lim_{x\to 0}\frac{1}{\sqrt{4+x}+2}=\frac{1}{4}.$$

$$(iii)\ \lim_{x\to 0+}\frac{\sin x}{\sqrt{x}}=\left(\lim_{x\to 0+}\frac{\sin x}{x}\right)\cdot\left(\lim_{x\to 0+}\sqrt{x}\right)=1\cdot 0=0.$$

Example 5.7. Evaluate $\lim\limits_{x\to 1}\dfrac{x^2-1}{x-1}$.

Solution. Let us evaluate the left hand and the right hand limits.

When $x\to 1-0$, put $x=1-h$, $h>0$.

Hence, $h\to 0+$ as $x\to 1-0$, so that

$$\lim_{x\to 1-0}\frac{x^2-1}{x-1}=\lim_{h\to 0+}\frac{(1-h)^2-1}{-h}=\lim_{h\to 0+}\frac{-h(2-h)}{-h}$$

$$=\lim_{h\to 0+}(2-h)=2$$

Again when $x\to 1+0$, put $x=1+h$, $h>0$.

$$\therefore\qquad \lim_{x\to 1+0}\frac{x^2-1}{x-1}=\lim_{h\to 0+}\frac{(1+h)^2-1}{h}=\lim_{h\to 0+}(2+h)=2.$$

So that both, the left hand and the right hand, limits exist and are equal. Hence limit of the given function exists and equals 2.

Note. Since $x \neq 1$, division by $(x-1)$ is permissible.

$$\therefore \qquad \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} (x+1) = 2.$$

Example 5.8. Evaluate $\lim_{x\to 0} \dfrac{e^{1/x}}{e^{1/x}+1}$.

Solution. Now when $x \to 0+$, $1/x \to \infty$, $e^{-1/x} \to 0$

and when $x \to 0-$, $1/x \to -\infty$, $e^{1/x} \to 0$.

$$\therefore \qquad \lim_{x\to 0+} \frac{e^{1/x}}{e^{1/x}+1} = \lim_{x\to 0+} \frac{1}{1+e^{-1/x}} = 1$$

and

$$\lim_{x\to 0-} \frac{e^{1/x}}{e^{1/x}+1} = \frac{0}{1} = 0$$

so that the left hand limit is not equal to the right hand limit.

Hence, $\lim_{x\to 0} \dfrac{e^{1/x}}{e^{1/x}+1}$ does not exist.

EXERCISE

Evaluate:

1. $\lim_{x\to 2} \dfrac{x^2-4}{x-2}$
2. $\lim_{x\to 0-} \dfrac{|x|}{x}$ and $\lim_{x\to 0+} \dfrac{|x|}{x}$
3. $\lim_{x\to 0} \dfrac{3x+|x|}{7x-5|x|}$
4. $\lim_{x\to 1} \dfrac{1}{(x-1)} \left(\dfrac{1}{x+3} - \dfrac{2}{3x+5} \right)$
5. $\lim_{x\to 0} \dfrac{1-2\cos x + \cos 2x}{x^2}$
6. Show that $\lim_{x\to 0} \dfrac{xe^{1/x}}{1+e^{1/x}} = 0$.

7. Show that $\lim_{x\to 0} \dfrac{e^{1/x}-1}{e^{1/x}+1}$ does not exist.

8. Show that $\lim_{x\to 0} \dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}$ does not exist.

9. If $\lim_{x\to c} f(x) = l$, then show that $\lim_{x\to c} |f(x)| = |l|$.

   [*Hint.* $\big||f(x)| - |l|\ \big| \le |f(x) - l|, \forall\, x$]

10. If $\lim_{x\to c} g(x) = m$ and $m \neq 0$, then show that $\lim_{x\to c} \dfrac{1}{g(x)} = \dfrac{1}{m}$.

ANSWERS

1. 4;
2. –1, 1;
3. Does not exist;
4. 1/32;
5. –1.

5.2 CONTINUOUS FUNCTIONS

Let f be a function defined on an interval $[a, b]$. We shall now consider the behaviour of f at points of $[a, b]$.

5.2.1 Continuity at a Point

Definition 1 (*Continuity at an internal point*). A function f is said to be continuous at a point c, $a < c < b$, if

$$\lim_{x\to c} f(x) = f(c)$$

In other words, the function is continuous at c, if for each $\varepsilon > 0, \exists\, \delta > 0$, such that

$$|f(x) - f(c)| < \varepsilon, \text{ when } |x - c| < \delta$$

Definition 2. A function f is said to be *continuous from the left* at c, if

$$\lim_{x\to c-0} f(x) = f(c)$$

Also f is *continuous from the right at* c, if

$$\lim_{x\to c+0} f(x) = f(c)$$

Clearly a function is continuous at c if and only if it is continuous from the left as well as from the right.

Definition 3 (*Continuity at an end point*). A function f defined on a closed interval $[a, b]$ is said to be continuous at the end point a if it is continuous from the right at a,

i.e.,
$$\lim_{x\to a+0} f(x) = f(a)$$

Also the function is continuous at the end point b of $[a, b]$ if

$$\lim_{x \to b-0} f(x) = f(b)$$

Thus a function f is continuous at a point c if

(*i*) $\lim_{x \to c} f(x)$ *exists, and*

(*ii*) *limit equals the value of the function at* $x = c$.

5.2.2 Continuity in an Interval

A function f is said to be continuous in an interval $[a, b]$ if it is continuous at every point of the interval.

5.2.3 Discontinuous Functions

A function is said to be *discontinuous* at a point c of its domain if it is not continuous there at. The point c is then called a *point of discontinuity* of the function.

Types of discontinuities

(*i*) A function f is said to have *a removable discontinuity* at $x = c$ if $\lim_{x \to c} f(x)$ exists but is not equal to the value $f(c)$ (which may or may not exist) of the function. Such a discontinuity can be removed by assigning a suitable value to the function at $x = c$.

(*ii*) f is said to have a *discontinuity of the first kind* at $x = c$ if $\lim_{x \to c-0} f(x)$ and $\lim_{x \to c+0} f(x)$ both exist but are not equal.

(*iii*) f is said to have a *discontinuity of the first kind from the left* at $x = c$ if $\lim_{x \to c-0} f(x)$ exists but is not equal to $f(c)$.

Discontinuity of the first kind from the right is similarly defined.

(*iv*) f is said to have a *discontinuity of the second kind* at $x = c$ if neither $\lim_{x \to c-0} f(x)$ nor $\lim_{x \to c+0} f(x)$ exists.

(*v*) f is said to have a *discontinuity of the second kind from the left* at $x = c$ if $\lim_{x \to c-0} f(x)$ does not exist.

Discontinuity of the second kind from the right may be defined similarly.

5.2.4 Theorems on Continuity

Theorem 5.2. *If f, g be two functions continuous at a point c, then the functions* $f + g$, $f - g$, fg *are also continuous at c and if* $g(c) \neq 0$, *then* f/g *is also continuous at c.*

The proof is left as an exercise.

Theorem 5.3. *A function f defined on an interval I is continuous at a point* $c \in I$ *if for every sequence* $\{c_n\}$ *in I converging to c, we have*

$$\lim_{n\to\infty} f(c_n) = f(c)$$

First let us suppose that the function f is continuous at a point $c \in I$, and $\{c_n\}$ is a sequence in I such that $\lim_{n\to\infty} c_n = c$.

Since f is continuous at c, therefore, for any given $\varepsilon > 0, \exists$ a $\delta > 0$, such that

$$|f(x) - f(c)| < \varepsilon, \text{ when } 0 < |x - c| < \delta \qquad ...(1)$$

Again, since $\lim_{n\to\infty} c_n = c$, therefore, $\exists$ a positive integer m, such that

$$|c_n - c| < \delta, \forall\ n \geq m \qquad ...(2)$$

From (1), putting $x = c_n$, we have

$$|f(c_n) - f(c)| < \varepsilon, \text{ when } |c_n - c| < \delta$$

$\Rightarrow$ $$|f(c_n) - f(c)| < \varepsilon, \quad \forall\ n \geq m \text{ [using 2]}$$

$\Rightarrow$ the sequence $\{f(c_n)\}$ converges to $f(c)$

or $$\lim_{n\to\infty} f(c_n) = f(c)$$

Let us now suppose that f is *not continuous* at c, we shall now show that though $\exists$ a sequence $\{c_n\}$ in I converging to c yet the sequence $\{f(c_n)\}$ does not converge to $f(c)$.

Since f is not continuous at c, therefore, there exists an $\varepsilon > 0$ such that for every $\delta > 0, \exists$ an $x \in I$, such that

$$|f(x) - f(c)| \geq \varepsilon, \text{ when } |x - c| < \delta$$

Also $$\lim c_n = c.$$

$\therefore$ By taking $\delta = 1/n$, we find that for each positive integer n, there is a $c_n \in I$, such that

$$|f(c_n) - f(c)| \geq \varepsilon, \text{ when } |c_n - c| < \frac{1}{n}$$

Thus, the sequence $\{f(c_n)\}$ does not converge to $f(c)$, while the sequence $\{c_n\}$ converges to c.

Notes. 1. If $\lim c_n = c \Rightarrow \lim f(c_n) \neq f(c)$, then f is not continuous at c.

2. Compare with § 1.3.

Example 5.9. Examine the following function for continuity at the origin:

$$f(x) = \begin{cases} \dfrac{xe^{1/x}}{1 + e^{1/x}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Now

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \frac{xe^{1/x}}{1 + e^{1/x}} = 0$$

and

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \frac{x}{e^{-1/x} + 1} = 0$$

Thus,

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x) = \lim_{x \to 0} f(x) = 0$$

Also

$$\lim_{x \to 0} f(x) = f(0)$$

Thus, the function is continuous at the origin.

Example 5.10. Show that the function defined as:

$$f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0 \end{cases}$$

has removable discontinuity at the origin.

Solution. Now

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin 2x}{2x} \cdot 2 = 2$$

so that

$$\lim_{x \to 0} f(x) \neq f(0)$$

Hence, the limit exists, but is not equal to the value of the function at the origin. Thus the function has a removable discontinuity at the origin.

Note. The discontinuity can be removed by re-defining the function at the origin such as $f(0) = 2$.

Example 5.11. Show that the function defined by

$$f(x) = \begin{cases} x \sin 1/x, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$

is continuous at $x = 0$.

Solution. Now

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(x \sin \frac{1}{x} \right) = 0$$

so that

$$\lim_{x \to 0} f(x) = f(0)$$

Hence, f is continuous at $x = 0$.

Example 5.12. A function f is defined on [0, 1] by

$$f(x) = \begin{cases} -x^2 & \text{if } x \le 0 \\ 5x - 4 & \text{if } 0 < x \le 1 \\ 4x^2 - 3x & \text{if } 1 < x < 2 \\ 3x + 4 & \text{if } x \ge 2 \end{cases}$$

Examine f for continuity at $x = 0, 1, 2$. Also discuss the kind of discontinuity, if any.

Solution. Now

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \left(-x^2\right) = 0$$

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} (5x - 4) = -4$$

so that

$$\lim_{x \to 0-} f(x) = f(0) \neq \lim_{x \to 0+} f(x)$$

Thus the function has a discontinuity of the first kind from the right at the origin.

Again

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} (5x - 4) = 1$$

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} \left(4x^2 - 3x\right) = 1$$

so that

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x) = 1 = f(1)$$

$$\Rightarrow \quad \lim_{x \to 1} f(x) = f(1)$$

Thus the function is continuous at $x = 1$.

Again

$$\lim_{x \to 2-} f(x) = \lim_{x \to 2-} \left(4x^2 - 3x\right) = 10$$

$$\lim_{x \to 2+} f(x) = \lim_{x \to 2+} (3x + 4) = 10$$

Also $f(2) = 10$

$$\Rightarrow \quad \lim_{x \to 2} f(x) = f(2)$$

Thus, the function is continuous at $x = 2$.

Example 5.13. Is the function f, where $f(x) = \dfrac{x - |x|}{x}$ continuous?

Solution. For $x < 0$, $f(x) = \dfrac{x + x}{2} = 2$, continuous.

For $x > 0$, $f(x) = \dfrac{x - x}{x} = 0$, continuous.

The function is not defined at $x = 0$.

Thus, $f(x)$ is continuous for all x except zero.

Example 5.14. Discuss the kind of discontinuity, if any of the function defined as follows:

$$f(x) = \begin{cases} \dfrac{x - |x|}{x}, & \text{when } x \neq 0 \\ 2, & \text{when } x = 0 \end{cases}$$

Solution. The function is continuous at all points except possibly the origin.

Let us test at $x = 0$.

Now

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \frac{x + x}{x} = 2$$

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \frac{x - x}{x} = 0$$

and

$$f(0) = 2$$

Thus the function has discontinuity of the first kind from the right at $x = 0$.

Example 5.15. If $[x]$ denotes the largest integer $\leq x$, then discuss the continuity at $x = 3$ for the function

$$f(x) = x - [x], \quad \forall\ x \geq 0$$

Solution. Now

$$\lim_{x \to 3-} f(x) = \lim_{x \to 3-} \{x - [x]\} = 3 - 2 = 1$$

$$\lim_{x \to 3+} f(x) = \lim_{x \to 3+} \{x - [x]\} = 3 - 3 = 0$$

and

$$f(3) = 0$$

Thus the function has a discontinuity of the first kind from the left at $x = 3$.

Note. The function is continuous at all points except the integral value 1, 2, 3, ...

Example 5.16. Prove that the *Dirichlet's* function f defined on **R** by

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is irrational} \\ -1, & \text{when } x \text{ is rational} \end{cases}$$

is discontinuous at every point.

Solution. First, let a be any *rational* number so that $f(a) = -1$.

Since in any interval there lie an infinite number of rational and irrational numbers, therefore, for each positive integer n, we can choose an irrational number a_n, such that $|a_n - a| < \dfrac{1}{n}$.

Thus, the sequence $\{a_n\}$ converges to a.

But $f(a_n) = 1$ for all n, and $f(a) = -1$, so that

$$\lim_{n \to \infty} f(a_n) \neq f(a)$$

Thus by Theorem 4, § 2.4, the function is discontinuous at any rational number a.

Hence the function is discontinuous at all rational points.

Next, let b be any *irrational* number. For each positive integer n we can choose a rational number b_n, such that $|b_n - b| < \frac{1}{n}$. Thus, the sequence $\{b_n\}$ converges to b.

But $f(b_n) = -1$ for all n and $f(b) = 1$.

$$\therefore \quad \lim_{n \to \infty} f(b_n) \neq f(b)$$

Hence, the function is discontinuous at all irrational points.

Example 5.17. Show that the function $f(x)$ defined on **R** by

$$f(x) = \begin{cases} x, & \text{when } x \text{ is irrational} \\ -x, & \text{when } x \text{ is rational} \end{cases}$$

is continuous only at $x = 0$.

Solution. First, let $a \neq 0$ be any *rational* number, so that $f(a) = -a$. Since in every interval there lie an infinite number of rational and irrational numbers, therefore, for each positive integer n, we can choose an irrational number a_n such that

$$|a_n - a| < \frac{1}{n}$$

Thus the sequence $\{a_n\}$ converges to a.

But

$$\lim_{n \to \infty} f(a_n) = \lim_{n \to \infty} a_n = a$$

Thus

$$\lim_{n \to \infty} f(a_n) \neq f(a), \; a \neq 0$$

so that by Theorem 5.3, § 2.4, the function is discontinuous at any rational number, other than zero.

In a similar way the function may be shown to be discontinuous at every *irrational* point.

It may be seen from above, that the function is continuous at $x = 0$ (*i.e.*, $a = 0$). However, it can be shown to be continuous at $x = 0$ as follows:

Let $\varepsilon > 0$ be given and let $\delta = \varepsilon$ (or any $\delta < \varepsilon$), then

$|x| < \delta \Rightarrow |f(x) - f(0)| = |-x| = |x| < \varepsilon$, when x is rational and

$|x| < \delta \Rightarrow |f(x) - f(0)| = |x| < \varepsilon$, when x is irrational.

Thus

$$|x| < \delta \Rightarrow |f(x) - f(0)| < \varepsilon$$

or

$$\lim_{x \to 0} f(x) = f(0)$$

Hence, the function is continuous at $x = 0$.

EXERCISE

Investigate the continuity at the indicated point:

1. $f(x)=\begin{cases}\dfrac{\sin(x-c)}{x-c}, & \text{if } x\neq c\\ 0, & \text{if } x=c\end{cases}$ at $x=c$

2. $f(x)=\begin{cases}\dfrac{\tan^{-1}x}{x}, & \text{if } x\neq 0\\ 1, & \text{if } x=0\end{cases}$ at $x=0$

3. $f(x)=\begin{cases}\dfrac{x^3-8}{x^2-4}, & \text{if } x\neq 2\\ 3, & \text{if } x=2\end{cases}$ at $x=2$

4. $f(x)=x-|x|$, at $x=0$

5. $f(x)=\begin{cases}x^m\sin\dfrac{1}{x}, & \text{if } x\neq 0, m>0\\ 0, & \text{if } x=0\end{cases}$ at $x=0$

6. $f(x)=\begin{cases}(1+x)^{1/x}, & \text{if } x\neq 0\\ 1, & \text{if } x=0\end{cases}$ at $x=0$

7. $f(x)=\begin{cases}\dfrac{e^{1/x^2}}{1-e^{1/x^2}}, & \text{when } x\neq 0\\ 1, & \text{when } x=0\end{cases}$ at $x=0$

8. $f(x)=\begin{cases}\dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}, & \text{when } x\neq 0\\ 1, & \text{when } x=0\end{cases}$ at $x=0$

9. Examine the continuity at $x=1$:

$$\begin{aligned} f(x) &= 2x, \text{ when } 0\le x<1\\ &= 3, \quad \text{when } x=1\\ &= 4x, \text{ when } 1<x\le 2\end{aligned}$$ [P.U.]

10. Obtain the points of discontinuity of the function f, defined on [0, 1] as follows:

$$f(0)=0,\ f(x)=\frac{1}{2}-x, \text{ if } 0<x<\frac{1}{2},$$

$$f\left(\frac{1}{2}\right)=\frac{1}{2},\ f(x)=\frac{3}{2}-x, \text{ if } \frac{1}{2}<x<1$$

$$f(1)=1$$

Also examine the kind of discontinuities. [P.U.]

11. Show that the function f defined on $\mathbf{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational,} \end{cases}$$

is discontinuous at every point.

12. Show that the function f defined by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is continuous only at $x = 0$.

ANSWERS

1. Removable discontinuity
2. Continuous
3. Continuous
4. Continuous
5. Continuous
6. Removable discontinuity
7. Removable discontinuity
8. Discontinuity of the first kind from the left
9. Discontinuity of the first kind
10. Discontinuity of the first kind from the right at 0, discontinuity of the first kind at $x = \frac{1}{2}$, discontinuity of the first kind from the left at $x = 1$.

5.3 FUNCTIONS CONTINUOUS ON CLOSED INTERVALS

We shall now study some properties of functions which are continuous on closed intervals. In fact, we shall show that a function which is continuous on a closed interval, is bounded, attains its bounds and assumes every value between the bounds.

Theorem 5.4. *If a function is continuous in a closed interval, then it is bounded therein.*

Let f be a function defined and continuous in a closed interval I.

We shall show that if the function f is not bounded, then it fails to be continuous at some point of the closed interval I.

Let, if possible, f be *not bounded above*, so that for each positive integer n $\exists$ a point $x_n \in I$ such that $f(x_n) > n$.

Now $\{x_n\}$, being a sequence in the closed interval I, is bounded and has at least one limit point, say ξ.

A closed interval is a closed set and so $\xi \in I$.

Further, since ξ is a limit point of the sequence $\{x_n\}$, therefore, there exists a subsequence (x_{n_k}) of $\{x_n\}$ such that $x_{n_k} \to \xi$ as $k \to \infty$.

Also since $f(x_{n_k}) > n_k$, for all k, therefore the sequence $\{f(x_{n_k})\}$ diverges to ∞.

Thus, $\exists$ a point ξ of I such that a sequence (x_{n_k}) in I converges to ξ, but

$$\lim_{k \to \infty} f\left(x_{n_k}\right) \neq f(\xi)$$

Thus, f is not continuous at ξ, which is a contradiction and hence the function is bounded above.

By considering a function $-f$, it can be shown in a similar way that the function f is also bounded below.

Hence, the function is bounded.

Theorem 5.5. *If a function f is continuous on a closed interval* [*a, b*], *then it attains its bounds at least once in* [*a*, *b*].

If *f* is a constant function, then evidently, it attains its bounds at every point of the interval.

Let *f* be a function which is not a constant.

Since *f* is continuous on the closed interval [*a*, *b*], therefore, it is bounded. Let *m* and *M* be the infimum and supremum of *f*. It is to be shown that $\exists$ points, α, β of [*a*, *b*] such that

$$f(\alpha) = m, \quad f(\beta) = M$$

Let us consider the case of the *supremum*.

Suppose *f* does not attain the supremum *M* so that the function does not take the value *M* for any point $x \in [a, b]$, *i.e.*,

$$f(x) \neq M, \text{ for any } x \in [a, b]$$

Now consider the function

$$g(x) = \frac{1}{M - f(x)}, \quad \forall\ x \in [a, b]$$

which is positive for all values of *x* in [*a*, *b*].

Evidently the function *g* is continuous and so bounded in [*a*, *b*].

Let *k* (> 0) be its supremum

$$\therefore \qquad \frac{1}{M - f(x)} \leq k, \quad \forall\ x \in [a, b]$$

$$\Rightarrow \qquad f(x) \leq M - \frac{1}{k}, \quad \forall\ x \in [a, b]$$

which contradicts the hypothesis that *M* is the supremum of *f* in [*a*, *b*]. Hence, our supposition that *f* does not attain the value *M* leads to a contradiction and therefore *f* attains its supremum for at least one value in [*a*, *b*].

It may similarly be shown that the function also attains its infimum *m*.

Hence, the function attains its bounds at least once in [*a*, *b*].

Note. It may be observed from the two preceding theorems, that the function *f*, continuous on the closed interval [*a*, *b*], has the least and the greatest values *m* and *M*, *i.e.*, the range set of *f* is bounded with *m* and *M* as its smallest and greatest elements. Thus the range set of *f* is a subset of [*m*, *M*]. We shall, in fact, show later that the range set of *f* is [*m*, *M*] itself and that *f* takes up every value between *m* and *M*.

ILLUSTRATIONS

1. The function $f(x) = \frac{1}{1 + |x|}$, for real *x*, is continuous and bounded and attains its supremum for $x = 0$ but does not attain the infimum.

2. The function $f(x) = -\dfrac{1}{1+|x|}$, for all $x \in \mathbf{R}$, is continuous and bounded, attains its infimum but not the supremum.
3. The function $f(x) = x$, for all $x \in]0, 1[$ is continuous and bounded but attains neither the infimum nor the supremum.

Theorem 5.6. *If a function f is continuous at an interior point c of an interval* [*a, b*] *and* $f(c) \neq 0$, *then* $\exists$ a $\delta > 0$ *such that f* (*x*) *has the same sign as f* (*c*), *for every* $x \in]c - \delta, c + \delta[$.

Since the function *f* is continuous at an interior point *c* of [*a*, *b*], therefore for any $\varepsilon > 0, \exists$ a $\delta > 0$, such that

$$|f(x) - f(c)| < \varepsilon, \qquad \forall\ x \in]c - \delta, c + \delta[$$

or

$$f(c) - \varepsilon < f(x) < f(c) + \varepsilon, \qquad \forall\ x \in]c - \delta, c + \delta[$$

When $f(c) > 0$, taking ε to be less than $f(c)$, we find that

$$f(x) > 0, \quad \forall\ x \in]c - \delta, c + \delta[$$

When $f(c) < 0$, taking ε to be less than $-f(c)$, we find that

$$f(x) < 0, \quad \forall\ x \in]c - \delta, c + \delta[$$

Hence the theorem.

Corrollary. If *f* is continuous at the end point *b* of [*a*, *b*] and $f(b) \neq 0$, then there exists an interval $]b - \delta, b]$, such that *f* (*x*) has the sign of *f* (*b*) for all *x* in $]b - \delta, b]$.

A similar result holds for continuity at *a*.

Note. When *c* is an interior point of the interval, the theorem may be restated as:
If a function f is continuous at an interior point c of an interval [*a, b*] *and* $f(c) \neq 0$, *then* $\exists$ *a neighbourhood N of c wherein f* (*x*) *has the same sign as f* (*c*), *for all* $x \in N$.

Theorem 5.7. *If a function f is continuous on a closed interval* [*a, b*] *and f* (*a*) *and f* (*b*) *are of opposite signs*, *then there exists at least one point* $\alpha \in [a, b]$ *such that f* (α) = 0.

Let us suppose that $f(a) > 0$ and $f(b) < 0$.

Let *S* consist of those points of [*a*, *b*] for which *f* (*x*) is positive, *i.e.*,

$$S = \{x : a \leq x \leq b \wedge f(x) > 0\}$$

Now

$$f(a) > 0 \Rightarrow a \in S \Rightarrow S \neq \phi$$

Also *S* is bounded by *a* and *b*.

$\therefore$ By the order compl eteness property, *S* has the supremum, say α, where $a \leq \alpha \leq b$.

We shall now show that

(*i*) $\alpha \neq a, \alpha \neq b,$ and

(*ii*) $f(\alpha) = 0.$

(*i*) First we show that $\alpha \neq a$

Since $f(a) > 0$, therefore $\exists$ a $\delta > 0$ such that

$$f(x) > 0, \quad \forall\ x \in [a, a + \delta[$$

$\Rightarrow \quad [a, a + \delta[\subseteq S$

$\Rightarrow$ the supremum α of S is greater than or equal to $a + \delta$

$\Rightarrow \alpha \neq a$

Now we shall show that $\alpha \neq b.$

Since $f(b) < 0$, therefore $\exists$ a $\delta > 0$ such that

$$f(x) < 0, \quad \forall\ x \in]b - \delta, b]$$

$\Rightarrow b - \delta$ is an upper bound of S

$\Rightarrow \alpha \leq b - \delta \Rightarrow \alpha \neq b$

(*ii*) We shall now show that $f(\alpha) \not> 0$ and $f(\alpha) \not< 0.$

If $f(\alpha) > 0,$ then $\exists$ a $\delta > 0$ such that

$$f(x) > 0, \ \forall\ x \in]\alpha - \delta, \alpha + \delta[$$

$\Rightarrow \quad]\alpha - \delta, \alpha + \delta[\subseteq S$

Let us choose a positive $\delta_2 < \delta$ such that $\alpha + \delta_2 \in]\alpha - \delta, \alpha + \delta[\Rightarrow$ a member $\alpha + \delta_2$ of S is greater than the supremum α of S, which is a contradiction.

$\therefore \quad f(\alpha) \not> 0$

Let now $f(\alpha) < 0,$ so that $\exists$ a $\delta_1 > 0$ such that

$$f(x) < 0, \ \forall\ x \in]\alpha - \delta_1, \alpha + \delta_1[\qquad ...(1)$$

Again, since α is the supremum of S, therefore, there exists a member β of S, where $\alpha - \delta_1 < \beta \leq \alpha$ such that

$$f(\beta) > 0$$

But from (1), $f(\beta) < 0,$ which is a contradiction.

$\therefore \quad f(\alpha) \not< 0$

Thus it follows that $f(\alpha) = 0.$

Theorem 5.8. *If a function f is continuous on* [*a, b*] *and* $f(a) \neq f(b),$ *then it assumes every value between f* (*a*) *and f* (*b*).

Let A be any number between $f(a)$ and $f(b)$. We shall show that there exists a number $c \in]a, b[$ such that $f(c) = A$.

Consider a function ϕ defined on [*a*, *b*] such that

$$\phi(x) = f(x) - A$$

Clearly $\phi(x)$ is continuous on [*a*, *b*].

Also

$$\phi(a) = f(a) - A, \text{ and } \phi(b) = f(b) - A$$

so that $\phi(a)$ and $\phi(b)$ are of opposite signs.

Thus the function ϕ is continuous on [*a*, *b*] and $\phi(a)$ and $\phi(b)$ are of opposite signs; therefore, by the previous theorem, $\exists\, c \in]a, b[$ such that

$$\phi(c) = 0$$

$$\Rightarrow \qquad f(c) - A = 0 \Rightarrow f(c) = A,$$

Corollary. A function *f*, which is continuous on a closed interval [*a*, *b*], assumes every value between its bounds.

Since the function *f* is continuous on the closed interval [*a*, *b*], therefore, it is bounded and attains its bounds on [*a*, *b*], *i.e.*, $\exists$ two numbers α, β in [*a*, *b*] such that

$$f(\alpha) = M \text{ and } f(\beta) = m,$$

where *M* and *m* are, respectively, the supremum and the infimum of *f*.

Since *f* is continuous on [*a*, *b*], therefore, it is continuous on $[\beta, \alpha]$ or $[\alpha, \beta]$ as the case may be, and consequently assumes every value between $f(\alpha)$ and $f(\beta)$.

Thus the function assumes every value between its bounds.

We may sum up in other words:

The range of a continuous function whose domain is a closed interval is as well a closed interval.

Or, in still better words:

The image of a closed interval under a continuous function (mapping) is a closed interval.

5.4 UNIFORM CONTINUITY

Let *f* be a function defined on an interval *I*. Then by definition, the function is continuous at any point $c \in I$ if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon, \text{ when } |x - c| < \delta.$$

For continuity at any other point $d \in I$, for the same ε, a $\delta_1 > 0$ would exist (not necessarily equal to δ). There is in fact a δ corresponding to each point of *I*. The number δ in general depends on the selection of ε and the point *c*. However, if a δ could be found which depends only on ε and not on the selection of the point *c*, such a δ would work for the whole interval *I* on which *f* is continuous. In such a case, *f* is said to be *uniformly continuous* on *I*. Thus, the notion of uniform continuity is *global* in character in as much as we talk of uniform continuity only on an interval.

The notion of continuity is, however, *local* in character in as much as we can talk of continuity at a point.

It may seem to a beginner that the infimum of the set consisting of δ's corresponding to different points of *I* would work for the whole of *I*. But the infimum may be zero. In general, therefore, a δ which

may work for the entire interval may not exist, so that every continuous function may not be uniformly continuous.

Definition. A function f defined on an interval I is said to be **uniformly continuous** on I if to each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x_2) - f(x_1)| < \varepsilon, \text{ for arbitrary points } x_1, x_2 \text{ of } I$$
$$\text{for which } |x_1 - x_2| < \delta$$

5.4.1 We shall now prove two theorems on uniform continuity.

Theorem 5.9. *A function which is uniformly continuous on an interval is continuous on that interval.*

Let a function f be uniformly continuous on an interval I, so that for a given $\varepsilon > 0$, there corresponds a $\delta > 0$ such that

$$|f(x_1) - f(x_2)| < \varepsilon, \text{ where } x_1, x_2 \text{ are any two points of } I \text{ for which}$$
$$|x_1 - x_2| < \delta$$

Let $x \in I$, then on taking $x_1 = x$, we find that for $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$|f(x) - f(x_2)| < \varepsilon, \text{ when } |x - x_2| < \delta.$$

Hence the function is continuous at every point $x_2 \in I$, *i.e.*, the function f is continuous on I.

Theorem 5.10. *A function which is continuous on a closed interval is also uniformly continuous on that interval.*

Let a function f be continuous on a closed interval I. Let, if possible, f be not uniformly continuous on I. Then there exists an $\varepsilon > 0$ such that for any $\delta > 0$, there are numbers $x, y \in I$ for which

$$|f(x) - f(y)| \not< \varepsilon, \text{ when } |x - y| < \delta$$

In particular for each positive integer n, we can find real numbers x_n, y_n in I such that

$$|f(x_n) - f(y_n)| \not< \varepsilon, \text{ when } |x_n - y_n| < 1/n \qquad ...(1)$$

Now $\{x_n\}$ and $\{y_n\}$ being sequences in the closed interval I, they are bounded and so each has at least one limit point, say ξ and η respectively.

As a closed interval is a closed set,

$$\therefore \qquad \xi \in I, \eta \in I$$

Further since ξ is a limit point of $\{x_n\}$, there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that $x_{n_k} \to \xi$.

Similarly, there exists a convergent subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to \eta$.

Again from (1), we find that

$$|f(x_{n_k}) - f(y_{n_k})| \not< \varepsilon, \text{ when } |x_{n_k} - y_{n_k}| < 1/n_k \le 1/k \qquad ...(2)$$

The second inequality shows that

$$\lim_{k \to \infty} x_{n_k} = \lim_{k \to \infty} y_{n_k}$$

$$\Rightarrow \qquad \xi = \eta$$

From the first inequality we find that in case the sequences $\{f(x_{n_k})\}$ and $\{f(y_{n_k})\}$ converge, the limits to which they converge are different.

We thus have two sequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ both of which converge to ξ but $\{f(x_{n_k})\}$ and $\{f(y_{n_k})\}$ do not converge to the same limit.

So by Theorem 4 § 2.4, f is not continuous at ξ, for, otherwise, the two sequences $\{f(x_{n_k})\}$ and $\{f(y_{n_k})\}$ would converge to the same point $f(\xi)$.

Thus we arrive at a contradiction and so the hypothesis that f is not uniformly continuous on I is false.

Hence f is uniformly continuous on I.

Example 5.18. Show that the function $f(x) = 1/x$ is not uniformly continuous on]0, 1].

Solution. Clearly the function is continuous on]0, 1].

It will be uniformly continuous on the given interval if for a given $\varepsilon > 0, \exists$ a $\delta > 0$, independent of the choice of points x and c in]0, 1], such that

$$\left|\frac{1}{x} - \frac{1}{c}\right| < \varepsilon, \text{ when } |x - c| < \delta$$

or

$$\left|\frac{c - x}{cx}\right| < \varepsilon, \text{ when } c - \delta < x < c + \delta \qquad ...(1)$$

If we take $c = \delta$, then the interval $]c - \delta, c + \delta[$ becomes $]0, 2\delta[$. Also condition (1) must hold for any x in this interval.

But

$$\frac{\delta - x}{\delta x} \to \infty \text{ as } x \to 0,$$

i.e., if we choose x sufficiently close to zero, then condition (1) is violated.

Hence $1/x$ is not uniformly continuous on]0, 1].

Example 5.19. Show that the function $f(x) = x^2$ is uniformly continuous on [–1, 1].

Solution. Let x_1, x_2 be any two points of [–1, 1], then

$$|f(x_1) - f(x_2)| = |x_1^2 - x_2^2| = |x_1 - x_2| \cdot |x_1 + x_2| < \varepsilon$$

$$\text{when } |x_1 - x_2| < \frac{1}{2}\varepsilon = \delta$$

(where δ is independent of the choice of x_1, x_2).

Thus for any $\varepsilon > 0, \exists$ a $\delta = \frac{1}{2}\varepsilon$ such that for any choice of x_1, x_2 in [–1, 1], we have

$$|f(x_1) - f(x_2)| < \varepsilon, \text{ when } |x_1 - x_2| < \frac{1}{2}\varepsilon = \delta$$

Thus the function f is uniformly continuous on [–1, 1].

EXERCISE

1. Show that the function f is uniformly continuous i.e., for any $\varepsilon > 0$, $\exists$ a δ independent of the choice of points in the given interval.
   (*i*) $f(x) = x^2$ in [1, 2],
   (*ii*) $f(x) = \sqrt{x}$ in $[0, 2]$
2. Show that the function $f(x) = x^2$ is not uniformly continuous on $[0, \infty[$.
3. If f and g are uniformly continuous on the same interval, prove that $f \pm g$ and fg are also uniformly continuous on the interval. State and prove an analogous theorem for f/g.

CHAPTER 6

Functions with Interval as Domain (II)

6.1 THE DERIVATIVE

In this chapter we shall study the derivative, its existence and applications. We shall be concerned mainly with the real valued functions of a real variable, *i.e.*, the domains and the ranges of the functions considered here will be sets of real numbers.

6.1.1 Derivability at a Point

Let f be a real valued function defined on an interval $I = [a, b] \subseteq \mathbf{R}$. It is said to be derivable at an *interior point* c (where $a < c < b$) if

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \text{ or } \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \text{ exists.}$$

The limit in case it exists, is called the *Derivative* or the *Differential Coefficient* of the function f at $x = c$, and is denoted by $f'(c)$. The limit exists when the left hand and the right hand limits exist and are equal.

$\lim_{x \to c-0} \frac{f(x) - f(c)}{x - c}$ is called the *Left hand Derivative* and is denoted by

$$f'(c-0),\ f'(c-) \text{ or } Lf'(c), \text{ while}$$

$\lim_{x \to c+0} \frac{f(x) - f(c)}{x - c}$ is called the *Right hand Derivative* and is denoted by

$$f'(c+0),\ f'(c+) \text{ or } Rf'(c).$$

Thus, the derivative $f'(c)$ exists when

$$Lf'(c) = Rf'(c)$$

6.1.2 Derivability in an Interval

A function f defined on $[a, b]$ is *derivable at the end point a, i.e.*, $f'(a)$ exists if,

$$\lim_{x \to a+0} \frac{f(x) - f(a)}{x - a} \text{ exists.}$$

In other words

$$f'(a) = \lim_{x \to a+0} \frac{f(x) - f(a)}{x - a}$$

Similarly, it is *derivable at the end point b,* if $\lim_{x \to b-0} \frac{f(x) - f(b)}{x - b}$ exists.

If a function is derivable at all points of an interval except the end points, it is said to be derivable in the *open interval.*

A function is derivable in the *closed interval* [*a, b*] if it is derivable in the open interval]*a, b*[and also at the end points *a* and *b.*

Example 6.1. Show that the function $f(x) = x^2$ is derivable on [0, 1].

Solution. Let x_0 be any point of]0, 1[, then

$$f'(x_0) = \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \to x_0} (x + x_0) = 2x_0$$

At the end points, we have

$$f'(0) = \lim_{x \to 0+0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{x^2}{x} = \lim_{x \to 0+} x = 0$$

$$f'(1) = \lim_{x \to 1-0} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1-} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1-} (x + 1) = 2$$

Thus the function is derivable in the closed interval [0, 1].

Example 6.2. A function *f* is defined on **R** by

$$f(x) = x, \text{ if } 0 \le x < 1$$
$$= 1, \text{ if } x \ge 1$$

Consider the derivability at $x = 1$.

Solution.

$$Lf'(1) = \lim_{x \to 1-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1-} \frac{x - 1}{x - 1} = 1$$

$$Rf'(1) = \lim_{x \to 1+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1+} \frac{1 - 1}{x - 1} = 0$$

$\therefore$ $Lf'(1) \neq Rf'(1)$

Thus, $\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$ does not exist, *i.e.*, $f'(1)$ does not exist.

Example 6.3. Consider the derivability of the function $f(x) = |x|$ at the origin.

Solution.

$$\text{Left hand derivative} = \lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0-} \frac{|x|}{x} = \lim_{x \to 0-} \frac{-x}{x} = -1$$

$$\text{Right hand derivative} = \lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0+} \frac{|x|}{x} = \lim_{x \to 0+} \frac{x}{x} = 1$$

Thus,

$$f'(0-) \neq f'(0+)$$

Hence, the function is not derivable at $x = 0$.

Example 6.4. A function f defined as:

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is derivable at $x = 0$ but $\lim_{x \to 0} f'(x) \neq f'(0)$.

Solution.

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin 1/x}{x}$$

$$= \lim_{x \to 0} (x \sin 1/x) = 0$$

From elementary calculus we know that for $x \neq 0$,

$$f'(x) = 2x \sin(1/x) - \cos(1/x)$$

Clearly $\lim_{x \to 0} f'(x)$ does not exist and therefore there is no possibility of $\lim_{x \to 0} f'(x)$ being equal to $f'(0)$.

Thus $f'(x)$ is not continuous at $x = 0$ but $f'(0)$ exists.

6.2 CONTINUOUS FUNCTIONS

In this section we shall consider a relation between derivability and continuity, viz.,

derivability at a point $\Rightarrow$ *continuity at that point*

Thus we shall prove that continuity at a point is a necessary condition for derivability at that point.

Theorem 6.1. *A function which is derivable at a point is necessarily continuous at that point.*

Let a function f be derivable at $x = c$.

Hence $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists.

Now

$$f(x) - f(c) = \frac{f(x) - f(c)}{(x - c)} (x - c), (x \neq c)$$

Taking limits as $x \to c$, we have

$$\lim_{x \to c} \{f(x) - f(c)\} = \lim_{x \to c} \left\{ \frac{f(x) - f(c)}{x - c} (x - c) \right\}$$

$$= \lim_{x \to c} \left\{ \frac{f(x) - f(c)}{x - c} \right\} \cdot \lim_{x \to c} (x - c)$$

$$= f'(c) \cdot 0 = 0$$

so that $\lim_{x \to c} f(x) = f(c)$, and therefore f is continuous at $x = c$.

6.2.1 It is to be clearly understood that while continuity is a necessary condition for derivability at a point, it is not a sufficient condition. We come across functions which are continuous at a point without being derivable thereat, and still many more functions may be constructed.

Consider the function f defined by

$$f(x) = |x|, \quad \forall\ x \in \mathbf{R}$$

$f(x)$ is continuous at $x = 0$, for

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0-} f(x) = 0 = f(0)$$

But as shown in example 6.3, $f'(0)$ does not exist. Thus the function is continuous but not derivable at the origin.

However, it was the genius of German mathematician Weierstrass, who gave a function which is continuous everywhere but not derivable anywhere, viz.,

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cos(3^n x), \quad \forall\ x \in \mathbf{R}$$

Proof, however, is beyond the scope of the present book.

Some Counter Examples

1. $f(x) = |x| + |x - 1|, \quad \forall\ x \in \mathbf{R}$

   Continuous but not derivable at $x = 0$ and $x = 1$.

2. $f(x) = |x - \alpha|$

   Continuous but not derivable at $x = \alpha$.

3. $f(x) = x \sin 1/x \qquad$ if $x \neq 0$

   $= 0 \qquad$ if $x = 0$

   Continuous but not derivable at the origin.

4. $f(x) = 0$ if $x \leq 0$

   $= x$ if $x > 0$

   Continuous but not derivable at $x = 0$.

6.2.2 The existence of the derivative of a function at a point depends on the existence of a limit, viz., $\lim_{x\to c} \dfrac{f(x)-f(c)}{x-c}$. Therefore, keeping in view the corresponding theorems on limits, one can easily establish the following fundamental **theorem on derivatives**.

If the functions f, g are derivable at c, then the functions $f+g$, $f-g$, fg *and* f/g $(g(c)\neq 0)$ *are also derivable at c, and*

$$(f\pm g)'(c) = f'(c)\pm g'(c)$$
$$(fg)'(c) = f'(c)g(c) + f(c)g'(c)$$
$$(f/g)'(c) = \{f'(c)g(c) - f(c)g'(c)\}\big/\{g(c)\}^2, \text{ if } g(c)\neq 0$$

To illustrate the procedure, we prove the following theorem.

Theorem 6.2. *If f is derivable at c and* $f(c)\neq 0$ *then the function* $1/f$ *is also derivable thereat, and*

$$(1/f)'(c) = -f'(c)/\{f(c)\}^2.$$

Since f is derivable at c, it is also continuous thereat. Again since $f(c)\neq 0$, there exists a neighbourhood N of c wherein f does not vanish.

Now

$$\frac{1/f(x)-1/f(c)}{x-c} = -\frac{f(x)-f(c)}{x-c}\cdot\frac{-1}{f(x)\,f(c)},\quad x\in N$$

Proceeding to limits when $x\to c$, we get

$$\left(\frac{1}{f}\right)(c) = \lim_{x\to c}\frac{1/f(x)-1/f(c)}{x-c}$$
$$= -f'(c)\cdot\frac{1}{f(c)\cdot f(c)} = -\frac{f'(c)}{\{f(c)\}^2}$$

Thus the limits exists and equals, $-f'(c)/\{f(c)\}^2$.

Note. If f and g be two functions having the same domain D and $f\pm g$ or fg be derivable at $c\in D$ then f and g are not necessarily derivable at c. Consider, for instance

(*i*) $f(x) = \begin{cases} 1/x & x\neq 0 \\ 0 & x=0 \end{cases}$ and $g(x) = -f(x)$.

$f+g$ is derivable at the origin but f and g are not.

(*ii*) $f(x) = \begin{cases} x\sin(1/x) & x\neq 0 \\ 0 & x=0 \end{cases}$ and $g(x) = x$

Here fg is derivable at the origin, whereas f is not.

(*iii*) $f(x) = |x|$ and $g(x) = -|x|$

$f(x)g(x) = -x^2$, so that fg is derivable at the origin but the functions f and g are not derivable.

Similarly many more functions may be constructed.

EXERCISE

1. Find the derivatives of the following functions at the indicated points:

   (*i*) $f(x) = K$, a constant, at $x = c$

   (*ii*) $f(x) = x$, at $x = 0$

   (*iii*) $f(x) = \sqrt{x}$, at $x = 4$

   (*iv*) $f(x) = e^x$, at $x = x_0$

2. Show that the function

$$f(x) = |x| + |x - 1|$$

   is derivable at all points except 0, 1.

3. $f(x) = \begin{cases} x^3 \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$

   Prove that $f(x)$ has a derivative at $x = 0$ and that $f(x)$ and $f'(x)$ are continuous at $x = 0$.

4. $f(x) = (x - a) \sin \dfrac{1}{x - a}$, $x \neq a$

   $= 0$, $x = a$

   Show that $f(x)$ is continuous but not derivable at $x = a$.

5. Discuss the derivability of the following functions:

   (*i*) $f(x) = \begin{cases} 2, & x \leq 1 \\ x, & x > 1 \end{cases}$, at $x = 1$

   (*ii*) $f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ x, & x > 1 \end{cases}$, at $x = 1$

   (*iii*) $f(x) = \begin{cases} 2x - 3, & 0 \leq x \leq 2 \\ x^2 - 3, & 2 < x \leq 4 \end{cases}$, at $x = 2, 4$

6. Show that the function $f(x) = x|x|$ is derivable at the origin.

7. Find the derivative of f at $x = 0$, where $f(x) = x^2 |x|$.

8. $f(x) = |x|$ and $g(x) = 3|x|$, $x \in R$,

   Show that f, g are not derivable at the origin but $\lim \left(f(x)/g(x)\right)$ exists and is equal to $\lim \left[f'(x)/g'(x)\right]$, when $x \to 0$.

9. Find $Lf'(0)$ and $Rf'(0)$ for the following functions:

(*i*) $f(x) = \begin{cases} x \tan^{-1} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(*ii*) $f(x) = \begin{cases} \dfrac{x(e^{1/x} - 1)}{(e^{1/x} + 1)}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

10. If $f(x) = \begin{cases} \dfrac{x(e^{1/x} - e^{-1/x})}{(e^{1/x} + e^{-1/x})}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Show that f is continuous but not derivable at $x = 0$ and $Lf'(0) = -1$, $Rf'(0) = 1$.

11. Examine the function f, where $f(x) = \left\{ \begin{matrix} x^m \sin 1/x, x \neq 0 \\ 0, \quad x = 0 \end{matrix} \right\}$ for derivability at the origin. Also determine m where f' is continuous at the origin.

ANSWERS

5. (*i*) Not derivable (*ii*) Not derivable, $Lf'(1) = 0$, $Rf'(1) = 1$.

(*iii*) Not derivable; at $x = 2$, $f'(4) = 8$.

7. 0. 9. (*i*) $-\pi/2, \pi/2$ (*ii*) $-1, 1$.

11. Derivable if $m > 1$; $m > 2$.

6.3 INCREASING AND DECREASING FUNCTIONS

A function f is said to be increasing or decreasing at a point $x = c$ according as the value of the function increases or decreases at that point with increase in x. Thus for any x in the neighbourhood $]c - \delta, c + \delta[$, $\delta > 0$ of c,

$f(c - \delta) \leq f(x) \leq f(c + \delta)$ for an increasing function,

$f(c - \delta) \geq f(x) \geq f(c + \delta)$ for a decreasing function.

A function is increasing or decreasing in the interval $[a, b]$ according as

$$f(x_2) \geq f(x_1) \text{ or } f(x_2) \leq f(x_1), \forall x_2 \geq x_1 \wedge x_1, x_2 \in [a, b].$$

The function is *strictly increasing* or *strictly decreasing* if the strict inequality holds in the above relations, *i.e.*,

$f(x_2) > f(x_1)$ for a strictly increasing function,

$f(x_2) < f(x_1)$ for a strictly decreasing function

$$\forall\ x_2 > x_1 \wedge x_1, x_2 \in [a, b].$$

A function is said to be *monotone* or *monotonic* in an interval I if it is either increasing in I or decreasing in I.

It is said to be *strictly monotone* in I if it is either strictly increasing or strictly decreasing in I.

6.3.1 Sign of the Derivative

Let c be **an interior point** of the domain $[a, b]$ of a function f and let $f'(c)$ exist and be positive.

By definition,

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c),\ x \neq c$$

Thus depending on the choice of a positive ε however small, $\exists$ a positive number δ such that

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \varepsilon, \text{ when } |x - c| < \delta,\ x \neq c$$

or

$$f'(c) - \varepsilon < \frac{f(x) - f(c)}{x - c} < f'(c) + \varepsilon, \text{ when } x \in]c - \delta, c + \delta[,\ x \neq c$$

If $\varepsilon > 0$ is selected less than $f'(c)$, $\dfrac{f(x) - f(c)}{x - c}$ lies between two positive numbers and is therefore itself positive.

Hence

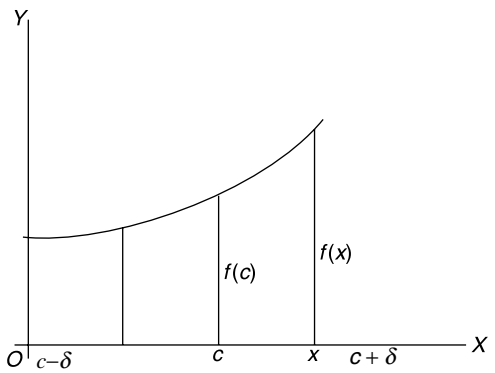
$$\frac{f(x) - f(c)}{x - c} > 0, \text{ when } x \in]c - \delta, c + \delta[,\ x \neq c.$$

Thus

(*i*) $f(x) - f(c) > 0$ or $f(x) > f(c)$ and $x \in]c, c + \delta[$, and

(*ii*) $f(x) < f(c)$, when $x \in]c - \delta, c[$.

From (*i*) and (*ii*) we see that $f(x)$ is increasing at $x = c$.



Hence, the function is increasing at c if $f'(c) > 0$.

Similarly, it can be shown that the function is decreasing at $x = c$ if $f'(c) < 0$.

Let us now consider the *end points.*

(a) At the *end point* a, $\exists$ an interval $[a, a + \delta[$, such that

$$f'(a) > 0 \Rightarrow f(x) > f(a), \text{ for } x \in]a, a + \delta[;$$

$$f'(a) < 0 \Rightarrow f(x) < f(a), \text{ for } x \in]a, a + \delta[.$$

(b) At the *end point* b, $\exists$ an interval $]b - \delta, b]$, such that

$$f'(b) > 0 \Rightarrow f(x) < f(b), \text{ for } x \in]b - \delta, b[;$$

$$f'(b) < 0 \Rightarrow f(x) > f(b), \text{ for } x \in]b - \delta, b[.$$

Example 6.5. Show that $\log (1 + x)$ lies between

$$x - \frac{x^2}{2} \text{ and } x - \frac{x^2}{2(1 + x)}, \ \forall\ x > 0.$$

Solution. Consider

$$f(x) = \log (1 + x) - \left(x - \frac{x^2}{2} \right)$$

$$\therefore \qquad f'(x) = \frac{1}{1 + x} - (1 - x) = \frac{x^2}{1 + x} > 0, \ \forall\ x > 0$$

Hence, $f(x)$ is an increasing function for all $x > 0$.

Also

$$f(0) = 0$$

Hence, for $x > 0$, $f(x) > 0$.

Thus

$$\log (1 + x) > x - \frac{x^2}{2}, \text{ for } x > 0$$

Similarly by considering the function

$$f(x) = x - \frac{x^2}{2(1 + x)} - \log (1 + x)$$

it can be shown that

$$\log (1 + x) < x - \frac{x^2}{2(1 + x)}, \ \forall\ x > 0.$$

EXERCISE

1. Show that

$$x^3 - 6x^2 + 15x + 3 > 0, \ \forall\ x > 0.$$

2. Show that

$$\frac{x}{1+x} < \log(1+x) < x, \qquad \forall\ x > 0\ .$$

3. Show that

(*i*) $\tan x < x, \qquad 0 < x < \pi/2$

(*ii*) $\frac{2}{\pi} \le \frac{\sin x}{x} < 1, \qquad 0 < |\pi| \le \pi/2,$

4. Show that

$$2x < \log \frac{1+x}{1-x} < 2x\left(1 + \frac{x^2}{3(1-x^2)}\right), 0 < x < 1.$$

6.4 DARBOUX'S THEOREM

If a function f is derivable on a closed interval [*a, b*] *and* $f'(a)$, $f'(b)$ *are of opposite signs then there exists at least one point c between a and b such that* $f'(c) = 0$.

For the sake of definiteness, let us take $f'(a) < 0$ and $f'(b) > 0$.

Since $f'(a)$ is negative, therefore, there exists a positive number δ_1 such that

$$f(x) < f(a), \quad \forall\ x \in]a, a + \delta_1[\qquad ...(1)$$

Again, since $f'(b)$ is positive, there exists a positive number δ_2 such that

$$f(x) < f(b), \quad \forall\ x \in]b - \delta_2, b[\qquad ...(2)$$

Also, since *f* is derivable in [*a*, *b*], it is continuous in the closed interval [*a*, *b*]. Being continuous in the closed interval, it is bounded and attains its bounds. Thus if *m* is the infimum (*g. l. b.*), $\exists$ a point $c \in [a, b]$ such that

$$f(c) = m$$

It is clear from (1) and (2) that *c* cannot coincide with *a* or *b* for otherwise there would exist points where the values of the function would be less than $f(c)$. Thus *c* is an interior point of [*a*, *b*].

Now we proceed to show that it is this point where $f'(c) = 0$.

If $f'(c) < 0$, then $\exists$ an interval $]c, c + \delta_3[$, $\delta_3 > 0$ such that $\forall\ x \in]c, c + \delta_3[$, $f(x) < f(c) = m$, which contradicts the fact that *m* is infimum of *f*.

Again, if $f'(c) > 0$, $\exists$ an interval $]c - \delta_4, c[$, $\delta_4 > 0$ for every point *x* of which

$f(x) < f(c) = m$, which again is a contradiction.

Hence, the only possibility, $f'(c) = 0$.

Note. If $f'(a) > 0$ and $f'(b) < 0$, then proceeding as above, it can be shown that $f'(d) = 0$ where $d \in]a, b[$ is the point where the function attains the supremum.

Corollary: If a function f is derivable on a closed interval $[a, b]$ and $f'(a) \neq f'(b)$ and k a number lying between $f'(a)$ and $f'(b)$ then $\exists$ at least one point $c \in]a, b[$ such that $f'(c) = k$.

Let $g(x) = f(x) - kx$, $x \in [a, b]$.

Clearly $g(x)$ is derivable on $[a, b]$ and

$$g'(a) = f'(a) - k,$$
$$g'(b) = f'(b) - k$$

Since, k lies between $f'(a)$ and $f'(b)$,

$\therefore$ $g'(a)$ and $g'(b)$ are of opposite signs.

$\therefore$ $g(x)$ satisfies the conditions of Darboux's theorem.

Thus, there exists at least one point $c \in]a, b[$ such that

$$g'(c) = 0 \text{ or } f'(c) = k.$$

6.5 ROLLE'S THEOREM

If a function f defined on $[a, b]$ *is*

(*i*) *continuous on* $[a, b]$,

(*ii*) *derivable on* $]a, b[$, *and*

(*iii*) $f(a) = f(b)$,

then there exists at least one real number c between a and b $(a < c < b)$ *such that* $f'(c) = 0$.

Since the function is continuous on the closed interval $[a, b]$ it is bounded and attains its bounds. Thus if m and M are the infimum (*g.l.b.*) and the supremum (*l.u.b.*) respectively of the function f then $\exists$ points c and d of $[a, b]$ such that

$$f(c) = m \text{ and } f(d) = M.$$

There are two possibilities: either $m = M$ or $m \neq M$.

If $m = M$, then f is constant over $[a, b]$ and therefore, its derivatives

$$f'(x) = 0, \quad \forall\ x \in [a, b].$$

When $m \neq M$, both of these cannot be equal to the same quantity $f(a)$. At least one of these, say, m is different from $f(a)$ or $f(b)$, so that

$$f(c) = m \neq f(a) \Rightarrow c \neq a$$
$$f(c) = m \neq f(b) \Rightarrow c \neq b$$

This means that c lies in the open interval $]a, b[$.

We shall now show that c is the point where $f'(c) = 0$.

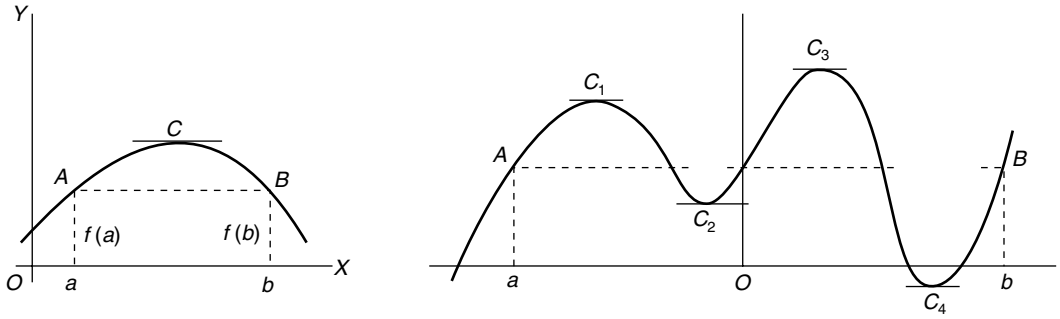
If $f'(c) < 0$, then $\exists$ an interval $]c, c + \delta_1[$, $\delta_1 > 0$ for every point x of which $f(x) < f(c) = m$; which contradicts the fact that m is the infimum.

If $f'(c) > 0$, $\exists$ an interval $]c - \delta_2, c[$, $\delta_2 > 0$ for every point x of which $f(x) < f(c) = m$; which is also a contradiction.

Hence the only possibility, $f'(c) = 0$.

6.5.1 Interpretation of Rolle's Theorem

Geometric. Let the curve $y = f(x)$, which is continuous on $[a, b]$ and derivable on $]a, b[$, be drawn.



The theorem simply states that between two points with equal ordinates on the graph of f, there exists at least one point where the tangent is parallel to x-axis.

Algebraic. Between two zeros a and b of $f(x)$ (*i.e.*, between two roots a and b of $f(x) = 0$) there exists at least one zero of $f'(x)$.

Exercise 1. Show that between two consecutive zeros of $f'(x)$ there lies at the most one zero of $f(x)$.

6.6 LAGRANGE'S MEAN VALUE THEOREM

(*First mean value theorem of differential calculus*)

If a function f defined on $[a, b]$, *is*

(*i*) *continuous on* $[a, b]$, *and*

(*ii*) *derivable on* $]a, b[$,

then there exists at least one real number c between a and b $(c \in]a, b[)$ *such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Let us consider a function

$$\phi(x) = f(x) + Ax,$$

where A is a constant to be determined such that $\phi(a) = \phi(b)$.

$$\therefore \qquad A = -\frac{f(b) - f(a)}{b - a}$$

Now the function $\phi(x)$, being the sum of two continuous and derivable functions, is itself

(*i*) continuous on $[a, b]$,

(*ii*) derivable on $]a, b[$, and

(*iii*) $\phi(a) = \phi(b)$.

Therefore, by Rolle's theorem $\exists$ a real number $c \in]a, b[$ such that $\phi'(c) = 0$.

But

$$\phi'(x) = f'(x) + A$$

$\therefore$ $$0 = \phi'(c) = f'(c) + A$$

or

$$f'(c) = -A = \frac{f(b) - f(a)}{b - a}$$

Another statement. If in the statement of the theorem, b is replaced by $a + h$, then the number c between a and b may be written as $a + \theta h$, where $0 < \theta < 1$. Thus,

$$f(a + h) - f(a) = hf'(a + \theta h)$$

or

$$f(a + h) = f(a) + hf'(a + \theta h), \text{ where } 0 < \theta < 1.$$

6.6.1 Deductions

1. If a function $f(x)$ satisfies the conditions of the Mean Value Theorem and $f'(x) = 0$ for all $x \in]a, b[$, then $f(x)$ is constant on $[a, b]$.

   Let x_1, x_2 (where $x_1 < x_2$) be any two distinct points of $[a, b]$.

   $\therefore$ By Lagrange's Mean Value Theorem

   $$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) = 0, \text{ where } x_1 < c < x_2 .$$

   Thus

   $$f(x_2) = f(x_1)$$

   The function keeps the same value and is therefore constant on $[a, b]$.
2. If two functions have equal derivatives at all points of $]a, b[$, then they differ only by a constant.
3. If a function f is (i) continuous on $[a, b]$, (ii) derivable on $]a, b[$, and (iii) $f'(x) > 0$, $\forall\ x \in]a, b[$, then f is strictly increasing on $[a, b]$.

   Let x_1, x_2 (where $x_1 < x_2$) be any two distinct points of $[a, b]$; then by Lagrange's Mean Value Theorem

   $$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0, \text{ for } x_1 < c < x_2 ,$$

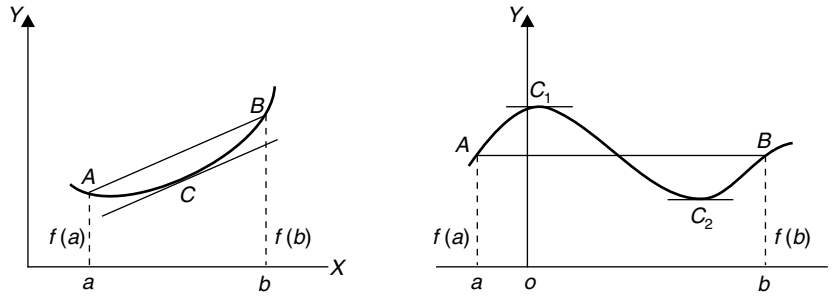
   or

   $$f(x_2) - f(x_1) > 0 \Rightarrow f(x_2) > f(x_1), \text{ for } x_2 > x_1 .$$

   Thus f is strictly increasing on $[a, b]$.

6.6.2 Geometrical Interpretation

The theorem simply states that between two points A and B of the graph of f there exists at least one point where the tangent is parallel to the chord AB.



6.7 CAUCHY'S MEAN VALUE THEOREM

(*Second mean value theorem*)

If two functions f, g defined on $[a, b]$ *are*

(*i*) *continuous on* $[a, b]$,

(*ii*) *derivable on* $]a, b[$, *and*

(*iii*) $g'(x) \neq 0$, *for any* $x \in]a, b[$,

then there exists at least one real number c between a and b $(c \in]a, b[)$ *such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

At the outset we notice that $g(a) \neq g(b)$, for otherwise, in view of the conditions (*i*) and (*ii*) of the theorem, $g(x)$ would satisfy all the conditions of Rolle's Theorem and its derivative $g'(x)$ would vanish for some $x \in]a, b[$, open interval contrary to (*iii*).

Consider the function

$$\phi(x) = f(x) + Ag(x),\ x \in [a, b],$$

where A is a constant to be determined such that $\phi(a) = \phi(b)$.

$$\therefore \qquad A = -\frac{f(b) - f(a)}{g(b) - g(a)}$$

Now the function $\phi(x)$, being the sum of two continuous and derivable functions, is itself.

(*i*) *continuous on* $[a, b]$,

(*ii*) *derivable on* $]a, b[$, *and*

(*iii*) $\phi(a) = \phi(b)$.

Therefore, by Rolle's Theorem $\exists$ a real number $c \in]a, b[$ such that $\phi'(c) = 0$

But $\phi'(x) = f'(x) + Ag'(x)$

$$\therefore \qquad 0 = \phi'(c) = f'(c) + Ag'(c)$$

$$\therefore \qquad \frac{f'(c)}{g'(c)} = -A = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Another statement. If two functions f, g defined on $[a, a+h]$ are continuous on $[a, a+h]$, derivable on $]a, a+h[$ and $g'(x) \neq 0$ for any $x \in]a, a+h[$, then there exists at least one real number θ between 0 and 1 such that

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)}, \quad 0<\theta<1.$$

Corollary. *Lagrange's Mean Value Theorem* may be deduced as a particular case, for $g(x) = x$.

Note. Cauchy's Mean Value Theorem cannot be deduced by applying Lagrange's Mean Value Theorem separately to the two functions and then dividing, for, then we get

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c_1)}{g'(c_2)}$$

where c_1 and c_2 may not be equal.

Geometrical (Physical) Interpretation. We may write

$$\frac{\{f(b)-f(a)\}/(b-a)}{\{g(b)-g(a)\}/(b-a)} = \frac{f'(c)}{g'(c)}$$

Hence, the ratio of the mean rates of increases of two functions in an interval is equal to the ratio of the actual rates of increases of the functions at some point within the interval.

Example 6.6. Show that

$$\frac{v-u}{1+v^2} < \tan^{-1}v - \tan^{-1}u < \frac{v-u}{1+u^2}, \text{ if } 0<u<v$$

and deduce that

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}.$$

Solution. Let $f(x) = \tan^{-1}x$;

$$f'(x) = \frac{1}{1+x^2}$$

Applying the Mean Value Theorem to f, we get

$$\frac{\tan^{-1}v - \tan^{-1}u}{v-u} = \frac{1}{1+\xi^2}, \text{ for } u<\xi<v$$

But

$$\xi > u \Rightarrow \frac{1}{1+\xi^2} < \frac{1}{1+u^2}$$

and

$$\xi < v \Rightarrow \frac{1}{1+\xi^2} > \frac{1}{1+v^2}$$

$$\therefore \quad \frac{1}{1+v^2} < \frac{\tan^{-1}v - \tan^{-1}u}{v-u} < \frac{1}{1+u^2}$$

or

$$\frac{v-u}{1+v^2} < \tan^{-1} v - \tan^{-1} u < \frac{v-u}{1+u^2}$$

The other result follows by taking $v = \frac{4}{3}$ and $u = 1$.

Example 6.7. Show that

$$\frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta, \text{ where } 0 < \alpha < \theta < \beta < \frac{\pi}{2}.$$

Let $f(x) = \sin x$ and $g(x) = \cos x$, for $x \in [\alpha, \beta]$.

$\therefore$ $$f'(x) = \cos x \text{ and } g'(x) = -\sin x$$

Solution. Functions f and g are both continuous and differentiable; therefore by Cauchy's Mean Value Theorem on $[\alpha, \beta]$,

$$\frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} = \frac{\cos\theta}{-\sin\theta}, \alpha < \theta < \beta$$

or

$$\frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta, \alpha < \theta < \beta.$$

Example 6.8. A twice differentiable function f is such that $f(a) = f(b) = 0$ and $f(c) > 0$, for $a < c < b$. Prove that there is at least one value ξ between a and b for which $f''(\xi) < 0$.

Solution. Let us consider the function f on $[a, b]$.

Since f'' exists, f' and f both exist and are continuous on $[a, b]$. Since c is a point between a and b, applying Lagrange's Mean Value Theorem to f on the intervals $[a, c]$ and $[c, b]$ respectively, we get

$$\frac{f(c) - f(a)}{c - a} = f'(\xi_1), \quad a < \xi_1 < c$$

and

$$\frac{f(b) - f(c)}{b - c} = f'(\xi_2), \quad c < \xi_2 < b$$

But

$$f(a) = f(b) = 0$$

$\therefore$ $$f'(\xi_1) = \frac{f(c)}{c - a} \text{ and } f'(\xi_2) = -\frac{f(c)}{b - c}$$

where $a < \xi_1 < c < \xi_2 < b$.

Again $f'(x)$ is continuous and derivable on $[\xi_1, \xi_2]$. Therefore, by Mean Value Theorem,

$$\frac{f'(\xi_2) - f'(\xi_1)}{\xi_2 - \xi_1} = f''(\xi), \text{ where } \xi_1 < \xi < \xi_2$$

Substituting the values of $f'(\xi_2)$ and $f'(\xi_1)$, we get

$$f''(\xi) = \frac{-f(c)}{\xi_2 - \xi_1}\left(\frac{1}{b-c} + \frac{1}{c-a}\right)$$

$$= \frac{-(b-a)f(c)}{(\xi_2 - \xi_1)(b-c)(c-a)} < 0.$$

Example 6.9. If a function f is such that its derivative f' is continuous on $[a, b]$ and derivable on $]a, b[$, then show that there exists a number c between a and b such that

$$f(b) = f(a) + (b-a)f'(a) = \frac{1}{2}(b-a)^2 f''(c).$$

Solution. Clearly the functions f and f' are continuous and derivable on $[a, b]$.

Consider the function

$$\phi(x) = f(b) - f(x) - (b-x)f'(x) - (b-x)^2 A$$

where A is a constant to be determined such that $\phi(a) = \phi(b)$.

$$\therefore \qquad f(b) - f(a) - (b-a)f'(a) - (b-a)^2 A = 0 \qquad \text{...(1)}$$

Now $\phi(x)$, being the sum of continuous and derivable functions, is itself continuous on $[a, b]$ and derivable on $]a, b[$ and also $\phi(a) = \phi(b)$.

Thus $\phi(x)$ satisfies all the conditions of Rolle's Theorem and therefore, $\exists\, c \in]a, b[$, such that $\phi'(c) = 0$.

Now

$$\phi'(x) = -f'(x) + f'(x) - (b-x)f''(x) + 2(b-x)A$$

$$\therefore \qquad -(b-c)f''(c) + 2(b-c)A = \phi'(c) = 0$$

But

$$b - c \neq 0$$

$$\therefore \qquad A = \frac{1}{2}f''(c) \qquad \text{...(2)}$$

Hence, from (1) and (2).

$$f(b) = f(a) + (b-a)f'(a) + \frac{1}{2}(b-a)^2 f''(c)$$

Note. Motivation for $\phi(x)$ is attained by replacing a by x and transposing all the terms to the right in the result to be proved.

Example 6.10. Assuming f'' to be continuous on $[a, b]$, show that

$$f(c) - f(a)\frac{b-c}{b-a} - \frac{c-a}{b-a}f(b) = \frac{1}{2}(c-a)(c-b)f''(\xi)$$

where c and ξ both lie in $[a, b]$.

Solution. We have to show that

$$(b-a)f(c)-(b-c)f(a)-(c-a)f(b)=\frac{1}{2}(b-a)(c-a)(c-b)f''(\xi)$$

Consider

$$\phi(x) = (b-a)f(x) - (b-x)\ f(a) - (x-a)f(b) - (b-a)(x-a)(x-b)\ A$$

where A is a constant to be determined such that $\phi(c) = 0$.

$$\therefore \qquad (b-a)f(c) - (b-c)\ f(a) - (c-a)f(b) - (b-a)(c-a)(c-b)\ A = 0 \qquad ...(1)$$

Clearly $\phi(a) = 0 = \phi(b)$, and $\phi(x)$ is differentiable in $[a, b]$.

The function ϕ satisfies all the conditions of Rolle's Theorem on each of the intervals $[a, c]$ and $[c, b]$ and therefore $\exists$ two numbers ξ_1, ξ_2 in $]a, c[$ and $]c, b[$, respectively, such that $\phi'(\xi_1) = 0$ and $\phi'(\xi_2) = 0$.

Again

$$\phi'(x) = (b-a)f'(x) + f(a) - f(b) - (b-a)\ \{2x - (a+b)\}A$$

which is continuous on $[a, b]$ and derivable on $]a, b[$ and in particular on $[\xi_1, \xi_2]$.

Also $\phi'(\xi_1) = \phi'(\xi_2) = 0$

Therefore by Rolle's Theorem $\exists\ \xi \in]\xi_1, \xi_2[$ such that $\phi''(\xi) = 0$.

Now $\phi''(x) = (b-a)\ f''(x) - 2(b-a)A$

so that

$$f''(\xi) - 2A = 0, b \neq a$$

or

$$A = \frac{1}{2}f''(\xi), \text{ where } a < \xi_1 < \xi < \xi_2 < b \qquad ...(2)$$

From (1) and (2) the result follows:

Example 6.11. Show that $\dfrac{\tan x}{x} > \dfrac{x}{\sin x}$, for $0 < x < \dfrac{\pi}{2}$.

Solution. We have thus to show that

$$\frac{\tan x}{x} - \frac{x}{\sin x} > 0 \quad \text{or} \quad \frac{\sin x \tan x - x^2}{x \sin x} > 0.$$

Since $x \sin x > 0$, for $0 < x < \pi/2$, it will therefore suffice to show that $\sin x \tan x - x^2 > 0$.

Let $f(x) = \sin x \tan x - x^2$

$f'(x) = \cos x \tan x + \sin x \sec^2 x - 2x = \sin x + \sin x \sec^2 x - 2x$.

We cannot decide about the sign of $f'(x)$ mainly because of the presence of the $2x$ term.

The function $f'(x)$ is continuous and derivable on $]0, \pi/2[$.

$$\therefore \qquad f''(x) = \cos x + \cos x \sec^2 x + 2\sin x \sec^2 x \tan x - 2$$

$$= \left(\sqrt{\sec x} - \sqrt{\cos x}\right)^2 + 2 \tan^2 x \sec x > 0, \text{ for } 0 < x < \pi/2$$

Since the derivative $f''(x)$ of $f'(x)$ is positive, the function $f'(x)$ is an increasing function. Further since $f'(0) = 0$, therefore the function $f'(x) > 0$ for $0 < x < \pi/2$.

Again, since $f'(x) > 0$, $f(x)$ is an increasing function and because $f(0) = 0$, the function $f(x) > 0$, for $0 < x < \pi/2$.

Thus it follows that

$$\frac{\tan x}{x} > \frac{x}{\sin x}, \text{ for } 0 < x < \pi/2.$$

EXERCISE

1. Examine the validity of the hypothesis and the conclusion of Rolle's Theorem:
   (*i*) $f(x) = x^3 - 4x$ on $[-2, 2]$,
   (*ii*) $f(x) = (x-a)^m (x-b)^n$, where m and n are positive integers on $[a, b]$,
   (*iii*) $f(x) = 1 - (x-1)^{2/3}$ on $[0, 2]$,
   (*iv*) $f(x) = |x|$ on $[-1, 1]$.
2. Examine the validity of the hypothesis and the conclusion of Lagrange's Mean Value Theorem:
   (*i*) $f(x) = |x|$ on $[-1, 1]$,
   (*ii*) $f(x) = \log x$ on $\left[\frac{1}{2}, 2\right]$,
   (*iii*) $f(x) = x(x-1)(x-2)$ on $\left[0, \frac{1}{2}\right]$
   (*iv*) $f(x) = x^{1/3}$ on $[-1, 1]$
   (*v*) $f(x) = 2x^2 - 7x + 10$ on $[2, 5]$.
3. Deduce Lagrange's Mean Value Theorem by considering the derivable function
   $$\phi(x) = f(x) - f(a) - A(x-a),\ x \in [a, b].$$
4. Prove that between any two real roots of $e^x \sin x = 1$, there is at least one real root of $e^x \cos x + 1 = 0$.
   [*Hint*: Apply Rolle's Theorem to the function $e^{-x} - \sin x$.]
5. A function f is continuous on $[a-h, a+h]$ and derivable on $]a-h, a+h[$, show that
   $$f(a-h) - 2f(a) + f(a+h) = h[f'(a+\theta h) - f'(a-\theta h)],\ 0 < \theta < 1$$
   [*Hint*: Use Mean Value Theorem for the function
   $$\phi(t) = f(a+th) + f(a-ht) \text{ on } [0, 1]].$$
6. If a function f is twice derivable on $[a, a+h]$, then show that
   $$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2} f''(a+\theta h),\ 0 < \theta < 1$$
7. If f', g' are continuous and differentiable on $[a, b]$, then show that for $a < c < b$
   $$\frac{f(b) - f(a) - (b-a)\, f'(a)}{g(b) - g(a) - (b-a)\, g'(a)} = \frac{f''(c)}{g''(c)}.$$

[*Hint*: Apply Rolle's Theorem to the function

$$\phi(x) = f(x) + (b - x)f'(x) + A\{g(x) + (b - x)g'(x)\}\,.$$

8. If f', ϕ and ψ are continuous on $[a, b]$ and derivable on $]a, b[$, then show that there is a value c lying between a and b such that

$$\begin{vmatrix} f(a) & f(b) & f'(c) \\ \phi(a) & \phi(b) & \phi'(c) \\ \psi(a) & \psi(b) & \psi'(c) \end{vmatrix} = 0.$$

[*Hint* : Apply Rolle's Theorem to the function

$$\phi(x) = \begin{vmatrix} f(a) & f(b) & f(x) \\ \phi(a) & \phi(b) & \phi(x) \\ \psi(a) & \psi(b) & \psi(x) \end{vmatrix}\Bigg]$$

9. A function f is such that its second derivative is continuous on $[a, a + h]$ and derivable on $]a, a + h[$. Show that there exists a number θ between 0 and 1 such that

$$f(a + h) - f(a) - \frac{1}{2}h\{f'(a) + f'(a + h)\} + \frac{h^3}{12}f''(a + \theta h) = 0.$$

10. If $f(0) = 0$ and $f''(x)$ exists for all $x > 0$, then show that

$$f'(x) - \frac{f(x)}{x} = \frac{1}{2}\,x f''(\xi)\,,\ 0 < \xi < x\,.$$

Also deduce that if $f''(x)$ is positive for positive values of x, then $f(x)/x$ strictly increases as x increases.

11. Apply Lagrange's Mean Value Theorem to the function $\log(1 + x)$ to show that

$$0 < [\log(1 + x)]^{-1} - x^{-1} < 1, \quad \forall\ x > 0\,.$$

12. Find the range of the values of x for which the function $x^3 - 6x^2 - 36x + 7$, increases with x.
13. Establish the following inequalities:

(*i*) $x - \dfrac{x^2}{2} + \dfrac{x^3}{3(1 + x)} < \log(1 + x) < x - \dfrac{x^2}{2} + \dfrac{x^3}{3}, \qquad x > 0$

(*ii*) $\dfrac{x^2}{2(1 + x)} < x - \log(1 + x) < \dfrac{x^2}{2}, \quad x > 0$

(*iii*) $\dfrac{x^2}{2} < x - \log(1 + x) < \dfrac{x^2}{2(1 + x)}, \quad -1 < x < 0$

(*iv*) $1 - x < -\log x < \dfrac{1}{x} - 1, \qquad 0 < x < 1$

(*v*) $x < -\log(1 - x) < x(1 - x)^{-1}, \qquad 0 < x < 1$

(*vi*) $x < \sin^{-1} x < \dfrac{x}{\sqrt{1 - x^2}}, \qquad 0 < x < 1$

(*vii*) $(1 - x) < e^{-x} < 1 - x + \dfrac{x^2}{2}, \qquad x > 0$

14. If $g(x) = 0$ has two equal roots, show that $g'(x) = 0$ has one root equal to either.

ANSWERS

1. (*i*), (*ii*) Both valid, (*iii*) and (*iv*) not valid.
2. (*i*) Not valid, (*ii*) valid, (*iii*) valid, $c = (6 - \sqrt{21})/6$,
   (*iv*) Hypothesis not valid but the conclusion is valid,
   (*v*) valid, $c = \frac{7}{2}$.
12. $x < -2$ and $x > 6$.

6.8 HIGHER ORDER DERIVATIVES

We know that the existence of the derivative f' of a function f at a point c implies the existence and continuity of the function in a neighbourhood of c. The derivative of the function f' at c in case it exists, is called the second derivative of f at c and denoted by $f''(c)$. Evidently the existence of $f''(c)$ implies the existence and continuity of f' in a neighbourhood of c.

Higher order derivatives can be similarly defined. The derivative of f^{n-1} at c in case it exists is called the nth derivative of f at c and is denoted by $f''(c)$.

6.8.1 Taylor's Theorem

If a function f defined on $[a, a + h]$*, is such that* (*i*) *the* $(n - 1)$*th derivative* f^{n-1} *is continuous on* $[a, a + h]$, *and* (*ii*) *the nth derivative* f^n *exists on* $]a, a + h[$, *then* $\exists$ *at least one real number* θ *between* 0 *and* $1 (0 < \theta < 1)$ *such that*

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \ldots$$

$$+ \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n (1 - \theta)^{n-p}}{p[(n-1)!]} f^n (a + \theta h) \qquad \ldots(1)$$

where p is a given positive integer.

First of all we observe that the condition (*i*) in the statement implies that all the derivatives $f', f'', \ldots, f^{n-1}$ exist and are continuous on $[a, a + h]$.

Consider the function ϕ defined on $[a, a + h]$ as

$$\phi(x) = f(x) + (a + h - x) f'(x) + \frac{(a + h - x)^2}{2!} f''(x) + \ldots$$

$$+ \frac{(a + h - x)^{n-1}}{(n-1)!} f^{n-1}(x) + A(a + h - x)^p$$

where A is a constant to be determined such that $\phi(a + h) = \phi(a)$.

$\therefore$ $$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots$$

$$+ \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + Ah^p \qquad ...(2)$$

Now,

(*i*) $f, f', f'', ..., f^{n-1}$ being all continuous on $[a, a+h]$, the function $\phi(x)$ is continuous on $[a, a+h]$;

(*ii*) the functions $f, f', ..., f^{n-1}$ and $(a+h-x)^r$ for all r being all derivable in $]a, a+h[$, the function $\phi(x)$ is derivable in $]a, a+h[$; and

(*iii*) $\phi(a+h) = \phi(a)$.

Thus, the function $\phi(x)$ satisfies all the conditions of Rolle's Theorem and hence $\exists$ at least one real number θ between 0 and 1 such that $\phi'(a+\theta h) = 0$.

But

$$\phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x) - Ap(a+h-x)^{p-1}$$

$$\therefore \quad 0 = \phi'(a+\theta h)$$

$$= \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h) - Aph^{p-1}(1-\theta)^{p-1}$$

$$\Rightarrow \quad A = \frac{h^{n-p}(1-\theta)^{n-p}}{p(n-1)!} f^n(a+\theta n), \quad h \neq 0, \theta \neq 1 \qquad ...(3)$$

Substituting A from (3) in (2), we get the required result.

Forms of Remainder after *n* Terms

(*i*) The term

$$R_n = \frac{h^n(1-\theta)^{n-p}}{p[(n-1)!]} f^n(a+\theta h)$$

which occurs after n terms, is known as Taylor's remainder after n terms. The theorem with this form of remainder is known as Taylor's Theorem with *Schlömilch and Röche form of remainder*.

(*ii*) For $p = 1$, we get

$$R_n = \frac{h^n(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h)$$

called *Cauchy's form of remainder*.

(*iii*) For $p = n$, we get

$$R_n = \frac{h^n}{n!} f^n(a+\theta h)$$

called *Lagrange's form of remainder*.

EXERCISE

1. Prove Taylor's Theorem with Lagrange's form of remainder by considering the function

$$\phi(x) = f(x) + (a + h - x)\, f'(x) + \frac{(a + h - x)^2}{2!} f''(x) + \ldots$$

$$\frac{(a + h - x)^{n-1}}{(n-1)!} f^{n-1}(x) + A(a + h - x)^n.$$

2. Prove Taylor's Theorem with Cauchy's form of remainder by taking the last term of $\phi(x)$ as $A(a + h - x)$.

Second form of Taylor's Theorem. If f satisfies the conditions of Taylor's Theorem in $[a, a + h]$, and x is any point of $[a, a + h]$ then it satisfies the conditions in the interval $[a, x]$ also.

Replacing $a + h$ by x or h by $(x - a)$ in (1), we get

$$f(x) = f(a) + (x - a) f'(a) + \frac{(x - a)^2}{2!} f''(a) + \ldots + \frac{(x - a)^{n-1}}{(n-1)!} f^{n-1}(a)$$

$$+ \frac{(x - a)^n}{p[(n-1)!]} (1 - \theta)^{n-p} f^n(a + \theta(x - a)) \qquad \ldots(4)$$

where $0 < \theta < 1$.

The remainder after n terms can thus be written as

$$R_n = \frac{(x - a)^n (1 - \theta)^{n-p}}{p[(n-1)!]} f^n(c)$$

where c lies between a and x and depends on the selection of x.

6.8.2 Maclaurin's Theorem

Putting $a = 0$ in (4), we have for $x \in]0, h[$,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0)$$

$$+ \frac{x^n (1 - \theta)^{n-p}}{p[(n-1)!]} f^n(\theta x) \qquad \ldots(5)$$

called Maclaurin's Theorem with *Schlömilch* and *Röche* form of remainder.

Cauchy's form of remainder (for $p = 1$):

$$R_n = \frac{x^n (1 - \theta)^{n-1}}{(n-1)!} f^n(\theta x).$$

Lagrange's form of remainder (for $p = n$):

$$R^n = \frac{x^n}{n!} f^n(\theta x).$$

We have thus proved **Maclaurin's Theorem**. Thus, Maclaurin's Theorem with Lagrange's form of remainder may be stated as:

If f^{n-1} *is continuous in* [0, *h*] *and is derivable in*]0, *h*[, *then for each* $x \in [0, h]$, *there exists a number* θ *between* 0 *and* 1 *such that*

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + ... + \frac{x^{n-1}}{(n-1)} f^{n-1}(0) + \frac{x^n}{n!} f''(\theta x).$$

6.8.3 Generalised Mean Value Theorem (*Taylor's theorem*)

Deduction of Taylor's Theorem from the Mean Value Theorem

Let a function *f* be such that $(n-1)$th derivative f^{n-1} is continuous in $[a, a+h]$ and its *n*th derivative f^n exists in $]a, a+h[$. Consequently, the functions $f, f', f'' \ldots f^{n-1}$ exist and are continuous in $[a, a+h]$ while f^n exists in $]a, a+h[$.

Consider the function

$$\phi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + ...$$

$$+ \frac{(a+h-x)^{n-1}}{(n-1)!} f^{n-1}(x)$$

which, being the sum of continuous and derivable functions, is itself continuous in $[a, a+h]$ and derivable in $]a, a+h[$. Therefore by Lagrange's Mean Value Theorem $\exists$ a positive number θ between 0 and 1, such that

$$\phi(a+h) = \phi(a) + h\phi'(a+\theta h)$$

Now

$$\phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x)$$

$$\therefore \qquad \phi'(a+\theta h) = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h)$$

Also

$$\phi(a) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a)$$

and

$$\phi(a+h) = f(a+h)$$

$$\therefore \qquad f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a)$$

$$+ \frac{h^n(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h)$$

where $0 < \theta < 1$, which is *Taylor's Theorem with Cauchy's form of remainder*.

Note. For $n = 1$, the theorem reduces to the Mean Value Theorem. For this reason Taylor's Theorem is also called General Mean Value Theorem.

6.8.4 Taylor's Infinite Series and Power Series Expansions

We have seen that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + R_n \qquad \ldots(6)$$

where R_n is the remainder after n terms.

The result can be interpreted in two ways:

(*i*) The value $f(a+h)$ of the function at a point may be approximated by a summation of the terms like $\frac{h^r}{r!}f^r(a)$ involving values of the function and its derivatives at some other point of the domain of definition.

(*ii*) The value $f(a+h)$ of the function may be expanded in powers of h.

The natural question as to how far the R.H.S. of (6) correctly represents the L.H.S. is answered if the series

$$f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots$$

converges to $f(a+h)$.

Let $S_n = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a)$, so that

$$f(a+h) = S_n + R_n$$

Thus if $R_n \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} S_n = f(a+h)$$

i.e., the infinite series

$$f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^n}{n!}f^n(a) + \ldots$$

converges to $f(a+h)$.

Thus we have proved that if a function f possesses derivatives of every order in $[a, a+h]$ and Taylor's remainder $R_n \to 0$ as $n \to \infty$, then

$$f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots = f(a+h)$$

The infinite series

$$f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \ldots \qquad \ldots(7)$$

is called **Taylor's series**. It can also be looked upon as expansion of $f(a+h)$ in powers of h.

Similarly for $x \in [a, a + h]$, when $\lim_{n\to\infty} R_n = 0$, we have from (4),

$$f(x) = f(a) + (x - a)\, f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots \qquad \ldots(8)$$

which is the expansion of $f(x)$ in powers of $(x - a)$.

6.8.5 Maclaurin's Infinite Series

We may easily deduce from (6) or (7) that if f possesses derivatives of every order in $[0, h[$ and $\lim_{n\to\infty} R_n = 0$, then for all $x \in [0, h]$,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots$$

which is *Maclaurin's infinite series* expansion of $f(x)$ in powers of x.

Note. In the above discussion the remainder R_n can be in any of the forms.

Example 6.12. Show that the number θ which occurs in the Taylor's Theorem with Lagrange's form of remainder after n terms approaches the limit $1/(n + 1)$ as h approaches zero, provided that $f^{n+1}(x)$ is continuous and different from zero at $x = a$.

Solution. Applying Taylor's Theorem with remainders after n terms and $n + 1$ terms successively, we get

$$f(a + h) = f(a) + hf'(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a + \theta h)$$

$$f(a + h) = f(a) + hf'(a) + \ldots + \frac{h^n}{n!} f^n(a) + \frac{h^{n+1}}{(n+1)!} f^{n+1}(a + \theta' h)$$

These give

$$\frac{h^n}{n!} f^n(a + \theta h) = \frac{h^n}{n!} f^n(a) + \frac{h^{n+1}}{(n+1)!} f^{n+1}(a + \theta' h)$$

or

$$f^n(a + \theta h) - f^n(a) = \frac{h}{n+1} f^{n+1}(a + \theta' h)$$

Applying Lagrange's Mean Value Theorem to the left hand side, we have

$$\theta h f^{n+1}(a + \theta'' \theta h) = \frac{h}{n+1} f^{n+1}(a + \theta' h)$$

or

$$\theta = \frac{1}{n+1} \frac{f^{n+1}(a + \theta' h)}{f^{n+1}(a + \theta'' \theta h)}$$

Taking the limit when $h \to 0$, we get

$$\lim_{h\to 0} \theta = \frac{1}{n+1}$$

6.8.6 To illustrate the applications of these theorems we consider series **expansion of the functions** e^x, $\cos x$, $\log (1 + x)$, $(1 + x)^m$ by Maclaurin's Theorem.

1. Let $f(x) = e^x$

   so that $f^n(x) = e^x$.

   Evidently $f(x)$ and all its derivatives exist and are continuous for every real value of x.

   Let us now consider the limit of the remainder R_n.

   Taking Lagrange's form of the remainder we have

$$R_n = \frac{x^n}{n!} f^n(\theta x) = \frac{x^n}{n!} e^{\theta x},\ 0 < \theta < 1$$

$$\therefore \qquad \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{x^n}{n!} e^{\theta x} = \left(\lim_{n\to\infty} \frac{x^n}{n!}\right) e^{\theta x} = 0 \quad \forall n$$

   Thus the conditions of Maclaurin's infinite expansion are satisfied. Now $f(0) = 1$ and $f^n(0) = 1$ for all integral values of n.

   Substituting these values in the Maclaurin's infinite series, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \qquad \forall\ x \in \mathbf{R}$$

2. Let $f(x) = \cos x$,

   so that $f^n(x) = \cos\left(\frac{1}{2} n\pi + x\right)$,

   Evidently $f(x)$ and all its derivatives exist and are continuous for every real value of x.

   Taking Lagrange's form of the remainder,

$$R_n = \frac{x^n f^n(\theta x)}{n!} = \frac{x^n}{n!} \cos\left(\frac{n\pi}{2} + \theta x\right)$$

$$\Rightarrow \qquad |R_n| = \left|\frac{x^n}{n!}\right| . \left|\cos\left(\frac{n\pi}{2} + \theta x\right)\right| \le \left|\frac{x^n}{n!}\right|$$

$$\therefore \qquad \lim_{n\to\infty} R_n = 0, \quad \forall\ x$$

   Thus the conditions of Maclaurin's infinite expansion are satisfied.

   Now

$$f(0) = 1,\ f'(0) = 0,\ f''(0) = -1, \ldots, f^n(0) = \cos(n\pi/2)$$

   which is zero for n odd, and alternately +1, –1 for n even.

   Substituting in the equation

$$f(x) = f(0) + x\ f'(0) + \frac{x^2}{2!} f''(0) + \ldots$$

we get

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \quad \forall \quad x \in \mathbf{R}$$

3. Let $f(x) = \log(1 + x)$, for $-1 < x \leq 1$; then

$$f^n(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}.$$

Evidently $f(x)$ and all its derivatives exist and are continuous for $|x| < 1$.

Taking the Lagrange's form of remainder, we have

$$R_n = \frac{x^n}{n!} f^n(\theta x) = \frac{(-1)^{n-1} x^n}{n(1+\theta x)^n} = (-1)^{n-1} \frac{1}{n}\left(\frac{x}{1+\theta x}\right)^n.$$

(*i*) When $0 \leq x \leq 1$,

$$0 < \frac{x}{1+\theta x} < 1$$

$$\therefore \qquad \left(\frac{x}{1+\theta x}\right)^n \to 0 \text{ as } n \to \infty$$

Also $\frac{1}{n} \to 0$ as $n \to \infty$

$$\therefore \qquad R_n \to 0 \text{ as } n \to \infty, \text{ for } 0 \leq x \leq 1.$$

Thus the conditions of Maclaurin's infinite series expansion are satisfied for $0 \leq x \leq 1$.

(*ii*) When $-1 < x < 0$.

In this case x may or may not be numerically less than $1 + \theta x$, so that nothing can be said about the limit of $\left(\frac{x}{1+\theta x}\right)^n$ when $n \to \infty$. Thus we fail to draw any definite conclusion from Lagrange's form of remainder. Let us now see if the other form of the remainder is of any help.

Cauchy's form of remainder,

$$R_n = \frac{x^n (1-\theta)^{n-1} f^n(\theta x)}{(n-1)!} = \frac{(-1)^{n-1} x^n (1-\theta)^{n-1}}{(1+\theta x)^n}$$

$$= (-1)^{n-1} x^n \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \cdot \frac{1}{(1+\theta x)}$$

Now $(1-\theta) < 1 + \theta x$, so that $\left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \to 0$ as $n \to \infty$.

Also

$$x^n \to 0 \text{ as } n \to \infty \text{ and } \frac{1}{1+\theta x} < \frac{1}{1-|x|}$$

and moreover it is independent of n.

Thus $R_n \to 0$ as $n \to \infty$.

Hence, the conditions of Maclaurin's series expansion are satisfied also when $-1 < x < 0$.

Thus substituting the values $f(0) = 0$, $f^n(0) = (-1)^{n-1}\,[(n-1)!]$ in Maclaurin's infinite expansion, we get

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, \text{ for } -1 < x \leq 1$$

4. Let $f(x) = (1+x)^m$.

Two cases arise according as m is or is not a positive integer.

(a) When m is a positive integer, $f(x)$ possesses continuous derivatives of all orders upto m. Derivatives of order higher than m all vanish identically. Consequently for $n > m$, $R_n \to 0$ identically so that the conditions of Maclaurin's expansion are satisfied. On substituting, the values of $f(0), f'(0), f''(0), \ldots, f^m(0)$, we get the finite expansion

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \ldots + x^m, \quad \forall\ x$$

(b) When m is not a positive integer, $(1+x)^m$ possesses continuous derivatives of all orders provided $x \neq -1$.

Let $-1 < x < 1$.

Taking Cauchy's form of remainder, we have

$$R_n = \frac{x^n}{(n-1)!}(1-\theta)^{n-1} f^n(\theta x)$$

$$= \frac{x^n}{(n-1)!}(1-\theta)^{n-1} m(m-1)\ldots(m-n+1)(1+\theta x)^{m-n}$$

$$= \left(\frac{m(m-1)\ldots(m-n+1)x^n}{(n-1)!}\right)\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}(1+\theta x)^{m-1}$$

We know for $|x| < 1$,

$$\frac{m(m-1)(m-2)\ldots(m-n+1)}{(n-1)!}x^n \to 0 \text{ as } n \to \infty,$$

and

$$\frac{1-\theta}{1+\theta x} < 1, \text{ so that } \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \to 0 \text{ as } n \to \infty.$$

Also

$$(1 + \theta x)^{m-1} < \left(1 + |x|\right)^{m-1},\ m > 1,\ 0 < \theta < 1$$

and

$$(1 + \theta x)^{m-1} = \frac{1}{(1 + \theta x)^{1-m}} < \frac{1}{\left(1 - |x|\right)^{1-m}},\ \text{when } m < 1$$

Thus $R_n \to 0$ when $n \to \infty$, for $|x| < 1$.

Hence, the conditions of Maclaurin's infinite expansion are satisfied.

Making the substitutions $f(0) = 1,\ f'(0) = m, \ldots, f^n(0) = m(m-1) \ldots (m-n+1)$, we get

$$(1 + x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)\,(m-2)}{3!}x^3 + \ldots \text{ for } |x| < 1.$$

Note. When m is not a positive integer the expansion is not possible if $|x| > 1$, for then as

$n \to \infty,\ \dfrac{m(m-1) \ldots (m-n+1)}{(n-1)!}$ and so R_n does not tend to zero.

EXERCISE

1. Expand, if possible, $\sin x$ in ascending powers of x.
2. Assuming the validity of expansion, show that

(*i*) $e^x \cos x = 1 + x - \dfrac{2x^3}{3!} - \dfrac{2^2x^4}{4!} - \dfrac{2^2x^5}{5!} + \ldots$

(*ii*) $\log \sec x = \dfrac{1}{2}x^2 + \dfrac{1}{12}x^4 + \ldots$

(*iii*) $\tan^{-1} x = \tan^{-1}\dfrac{\pi}{4} + \dfrac{x - \pi/4}{1 + \pi^2/16} - \dfrac{\pi(x - \pi/4)^2}{4(1 + \pi^2/16)^2} + \ldots$

(*iv*) $\sin\left(\dfrac{\pi}{4} + \theta\right) = \dfrac{1}{\sqrt{2}}\left(1 + \theta - \dfrac{\theta^2}{2!} - \dfrac{\theta^3}{3!} + \ldots\right)$

(*v*) $f(x) = f(a) + 2\left[\dfrac{x-a}{2} f'\left(\dfrac{x+a}{2}\right) + \dfrac{(x-a)^3}{8.(3)!} f'''\left(\dfrac{x+a}{2}\right)\right.$

$\left. + \dfrac{(x-a)^5}{32(5)!} f^{(5)}\left(\dfrac{x+a}{2}\right) + \ldots\right].$

3. Use Taylor's theorem to show that

   (*i*) $\cos x \geq 1 - \dfrac{x^2}{2}$, for all real x.

   (*ii*) $x - \dfrac{x^3}{6} < \sin x < x$, for $x > 0$

   (*iii*) $x - \dfrac{x^3}{6} < \sin x < x - \dfrac{x^3}{6} + \dfrac{x^5}{120}$, $\quad \forall\ x$

   (*iv*) $1 + x + \dfrac{x^2}{2} < e^x < 1 + x + \dfrac{x^2}{2}\, e^x$, $\quad x > 0$.

CHAPTER 7

Applications of Taylor's Theorem

MAXIMA AND MINIMA, INDETERMINATE FORMS

In this chapter we shall discuss applications of Taylor's theorem to two types of problems: (*a*) extreme values of a function, and (*b*) the evaluation of certain limits, popularly known as Indeterminate Forms.

7.1 EXTREME VALUES (*Definitions*)

Let c be an interior point of the domain $[a, b]$ of a function f.

Definition 1. The point c is said to be the *stationary point* and $f(c)$ the *stationary value* of the function f if $f'(c) = 0$.

Definition 2. The function f is said to have a **maximum value** (a *maxima* or a *maximum*) at c if $f(c)$ is the greatest value of the function in a small neighbourhood $]c - \delta, c + \delta[, \delta > 0,$ of c.

Thus for all $x \in]c - \delta, c + \delta[, x \neq c,$ we have

$$f(c) > f(x)$$

$\Rightarrow f(x) - f(c)$ is negative, for all values of x in $]c - \delta, c + \delta[$ other than c.

Definition 3. $f(c)$ is said to be the **minimum value**, a *minimum* or a *minima* if $f(c)$ is the least value of the function in a small neighbourhood $]c - \delta, c + \delta[$ of c. Thus, for all $x \in]c - \delta, c + \delta[, x \neq c,$

$$f(c) < f(x)$$

$\Rightarrow f(x) - f(c)$ is positive, for all values of x in a deleted neighbourhood $]c - \delta, c + \delta[$ of c.

Definition 4. The function f is said to have an *extreme value* at c if $f(c)$ is either a maximum or a minimum value. Thus at an extreme point c, $f(x) - f(c)$ keeps the same sign, for all values of x in a deleted neighbourhood $]c - \delta, c + \delta[$ of c.

7.1.1 A Necessary Condition for Extreme Values

Theorem 7.1. *If $f(c)$ is an extreme value of a function f then $f'(c)$, in case it exists, is zero.*

For the sake of definiteness, let us assume that $f(c)$ is a maximum value.

$\therefore \exists$ a $\delta > 0$ such that

$$f(x) < f(c), \ \forall \ x \in]c - \delta, c + \delta[, x \neq c$$

In case $f'(c)$ exists, there are three possibilities.

$$f'(c) > 0, \ f'(c) < 0, \ f'(c) = 0.$$

If $f'(c) > 0$, then $\exists$ an interval $]c, c + \delta_1[$, $0 < \delta_1 < \delta$ for every point of which $f(x) > f(c)$, which contradicts the fact that $f(c)$ is a maximum value.

Again if $f'(c) < 0$, then $\exists$ an interval $]c - \delta_2, c[$, $0 < \delta_2 < \delta$ for every point of which $f(x) > f(c)$ which again is a contradiction.

Hence, the only possibility, $f'(c) = 0$.

Remarks 1. The above theorem states that if the derivative exists, it must vanish at the extreme value. A function may however have an extreme value at a point without being derivable thereat. Consider, for example, the function $f(x) = |x|$, which has a minimum at the origin even through $f'(0)$ does not exist.

2. The vanishing of the derivative at a point is only a necessary condition for the existence of an extreme value; it is not sufficient. Functions exist for which the derivative vanish at a point but do not have an extreme value thereat *e.g.*, $f(x) = x^3$ at $x = 0$, so that the stationary points are not necessarily points of extreme values.

7.1.2 Investigation of the Points of Maximum and Minimum Values

At a point of extreme value the derivative of the function either does not exist or in case it exists, it must vanish.

Let c be an extreme point of a function f with domain $[a, b]$. If c is a *point of maximum value*, then $\exists$ a neighbourhood $]c - \delta, c + \delta[$ of c such that

$$f(x) < f(c), \quad \forall\ x \in\ ]c - \delta, c + \delta[,\ x \neq c$$

$$\therefore \qquad f(x) < f(c), \quad \forall\ x \in\ ]c - \delta, c[$$

$\Rightarrow$ f is increasing (to $f(c)$) in a small interval to the left of c.

Again

$$f(x) < f(c), \quad \forall\ x \in\ ]c, c + \delta[$$

$\Rightarrow$ f is decreasing (from $f(c)$) in a small interval to the right of c.

We may therefore state:

c is a point of maximum value if the function changes from an increasing to a decreasing function as x passes through c. Therefore, in case f is derivable, the derivative changes sign from positive to negative as x passes through c.

It can similarly be seen that c is a point of *minimum value* if f changes from a decreasing to an increasing function and in case f is derivable, the derivative f' changes sign from negative to positive as x passes through c.

Notes 1. It appears that f' is a decreasing (increasing) function in a neighbourhood of the point of maxima (minima) and therefore the second derivative f'' in case it exists, it would be negative (positive) as such a point.

2. The above conditions are sufficient but not necessary.

7.1.3 Before discussing the subject further, let us investigate the extreme values in a few cases.

Example 7.1. Examine the function $(x - 3)^5 (x + 1)^4$ for extreme values.

Solution. Let $f(x) = (x-3)^5 (x+1)^4$.

$\therefore$ $$f'(x) = (x-3)^4 (x+1)^3 (9x-7)$$

The function is derivable for all $x \in \mathbf{R}$ and the derivative f' vanishes for $x = -1, 3, \frac{7}{9}$ which may now be tested for extreme values.

(*a*) $x = -1$

f' is positive for a value of x slightly less than –1, and negative for slightly greater than –1.

Thus f' changes sign from + to – as x passes through –1.

Hence –1 is a point of maximum value.

(*b*) $x = 3$

f' remains positive as x passes through 3.

Hence $x = 3$ is neither a maxima nor a minima.

(*c*) $x = \frac{7}{9}$

Since f' changes from – to + as x passes through $\frac{7}{9}$, therefore f has a minimum value at $x = \frac{7}{9}$.

Example 7.2. Prove that a conical tent of a given capacity will require the least amount of canvas when the height is $\sqrt{2}$ times the radius of the base.

Solution. Let the tent be a cone of semi-vertical angle α and radius of the base r. The volume V is fixed, while the surface area S has to be the minimum.

Now

$$V = \frac{1}{3}\pi r^3 \cot \alpha \qquad ...(1)$$

and

$$S = \pi r^2 \operatorname{cosec} \alpha \qquad ...(2)$$

Differentiating (1), we get

$$\frac{\pi}{3}\left[3r^2 \cot \alpha \frac{dr}{d\alpha} - r^3 \operatorname{cosec}^2 \alpha\right] = 0$$

or $$\frac{dr}{d\alpha} = \frac{r \operatorname{cosec}^2 \alpha}{3 \cot \alpha}$$

Again from (2)

$$\frac{dS}{d\alpha} = \pi\left[2r \operatorname{cosec} \alpha \frac{dr}{d\alpha} - r^2 \cot \alpha \operatorname{cosec} \alpha\right]$$

$$= \frac{\pi r^2 \operatorname{cosec} \alpha}{3 \cot \alpha}\left[2 \operatorname{cosec}^2 \alpha - 3 \cot^2 \alpha\right]$$

$$= \frac{\pi r^2}{3 \cos \alpha}\left[2 - \cot^2 \alpha\right]$$

$$\therefore \qquad \frac{dS}{d\alpha} = 0, \text{ when } \cot\alpha = \sqrt{2}$$

$$\Rightarrow \qquad \alpha = \cot^{-1}\sqrt{2}.$$

Also $dS/d\alpha$ changes sign from negative to positive as α passes through the value $\cot^{-1}\sqrt{2}$.

$\therefore$ S has a minimum value at $\alpha = \cot^{-1}\sqrt{2}$.

Hence the height of the tent $= r \cot\alpha = r\sqrt{2}$

$= \sqrt{2}$ times the radius of the base.

7.1.4 In most cases the conclusion of § 7.1 suffice to determine the points of extreme values but to simplify the matters, recourse may be had to higher derivatives. In that context the following theorem will prove very useful.

Theorem 7.2. *If c is an interior point of the domain of a function f and $f'(c) = 0$, then the function has a maxima or a minima at c according as $f''(c)$ is negative or positive.*

The existence of $f''(c)$ implies that f and f' exist and are continuous at c. Continuity at c implies the existence of f and f' in a certain neighbourhood $]c-\delta, c+\delta[\,, \delta > 0$, of c, the neighbourhood being itself contained in the domain of f.

Let $f''(c) > 0$.

The implies that $f'(x)$ is an increasing function of c.

$$\Rightarrow \qquad f'(x) > f'(c) = 0, \quad \forall\ x \in]c, c+\delta_1[\,, \delta_1 < \delta$$

and

$$f'(x) < f'(c) = 0 \quad \forall\ x \in]c-\delta_1, c[$$

Thus first implies that $f'(x)$ is positive and hence f is an increasing function in $]c, c+\delta_1[$, *i.e.*, $f(x) > f(c)$ in $]c, c+\delta_1[$.

Similarly $f(x) > f(c)$ in $]c-\delta_1, c[$.

The last two results imply that

$$f(x) > f(c), \forall\ x \in]c-\delta_1, c+\delta_1[\,, x \neq c.$$

$\Rightarrow$ f has a minima at c.

Similarly $f''(c) < 0 \Rightarrow f(x)$ has a maxima at c.

Example 7.3. Examine the function $\sin x + \cos x$ for extreme values.

Solution. Let

$$f(x) = \sin x + \cos x$$

$$f'(x) = \cos x - \sin x$$

$$f''(x) = -\sin x - \cos x$$

$f'(x) = 0$ when $\tan x = 1$, so that

$$x = n\pi + \frac{\pi}{4}$$

where n is zero or any integer.

$$f''\left(n\pi + \frac{1}{4}\pi\right) = -\left\{\sin\left(n\pi + \frac{1}{4}\pi\right) + \cos\left(n\pi + \frac{1}{4}\pi\right)\right\}$$

$$= (-1)^{n+1}\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) = (-1)^{n+1}\sqrt{2}\,.$$

Also

$$f\left(n\pi + \frac{1}{4}\pi\right) = \sin\left(n\pi + \frac{1}{4}\pi\right) + \cos\left(n\pi + \frac{1}{4}\pi\right) = (-1)^n\sqrt{2}$$

When n is zero or an even integer, $f''(n\pi + \pi/4)$ is negative and therefore $x = n\pi + \frac{1}{4}\pi$ makes $f(x)$ a maxima with the maximum value $\sqrt{2}$.

When n is an odd integer, $f''\left(n\pi + \frac{1}{4}\pi\right)$ is positive and therefore $x = n\pi + \frac{1}{4}\pi$ makes $f(x)$ a minima with the minimum value $-\sqrt{2}$.

7.1.5* In cases where the second derivative vanishes, the above theorem fails to give any result. In those cases, we make use of still higher derivatives, and the following theorem proves very helpful.

General Theorem. *If c is an interior point of the domain $[a, b]$ of a function f and is such that*

(*i*) $f'(c) = f''(c) = f'''(c) = \ldots = f^{n-1}(c) = 0$, *and*

(*ii*) $f^n(c)$ *exists and is not zero,*

then for n odd, $f(c)$ is not an extreme value, while for n even $f(c)$ is a maximum or minimum value according as $f^n(c)$ is negative or positive.

Condition (*ii*) of the existence of $f^n(c)$ implies that $f, f', f'', \ldots, f^{n-1}$ all exist and are continuous at c. Also continuity at c implies the existence of $f, f', \ldots, f^{n-1}$ in a certain neighbourhood $]c - \delta_1, c + \delta_1[$ of c, $(\delta_1 > 0)$.

As $f^n(c) \neq 0$ there exists a neighbourhood $]c - \delta, c + \delta[$, $(0 < \delta < \delta_1)$ such that for $f^n(c) > 0$,

$$\left.\begin{aligned} f^{n-1}(x) < f^{n-1}(c) = 0, &\qquad x \in]c - \delta, c[\\ f^{n-1}(x) > f^{n-1}(c) = 0, &\qquad x \in]c, c + \delta[\end{aligned}\right\} \quad \ldots(1)$$

and

and for $f^n(c) < 0$,

$$\left.\begin{aligned} f^{n-1}(x) > f^{n-1}(c) = 0, &\qquad x \in]c - \delta, c[\\ f^{n-1}(x) < f^{n-1}(c) = 0, &\qquad x \in]c, c + \delta[\end{aligned}\right\} \quad \ldots(2)$$

and

Again for any real number h where $|h| < \delta$ we have by Taylor's Theorem

$$f(c + h) - f(c) + hf'(c) + \frac{h^2}{2!}f''(c) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(c + \theta h),\ 0 < \theta < 1$$

or $$f(c+h) - f(c) = \frac{h^{n-1}}{(n-1)!} f^{n-1}(c+\theta h) \qquad ...(3)$$

where $c + \theta h \in]c - \delta, c + \delta[$.

For n odd: Clearly $h^{n-1} > 0$ for any real number h, and further:

(*i*) When $f^n(c) > 0$, we deduce from (1) that for h negative, $(c + \theta h)$ is in $[c - \delta, c[$, and so $f^{n-1}(c + \theta h) < 0$, and for h positive, $f^{n-1}(c + \theta h) > 0$.

Thus from (3)

$$f(c+h) < f(c), \text{ when } c - \delta < c + h < c$$

and

$$f(c+h) > f(c), \text{ when } c < c + h < c + \delta$$

Thus $f(c)$ is not an extreme value.

(*ii*) When $f^n(c) < 0$, it may similarly be shown that $f(c)$ is not an extreme value.

For n even: As h^{n-1} is positive or negative according as h is positive or negative, we deduce as before from (1) and (3) that if $f^n(c) > 0$ then for every point $x = c + h \in]c - \delta, c + \delta[$ except c,

$$f(c+h) > f(c)$$

i.e., $f(c)$ is a minimum value.

It may similarly be deduced from (1) and (3) that $f(c)$ is a maximum value if $f^n(c) < 0$.

Notes 1. As a consequence of the above theorem, if f' vanishes at c, then c is a point of maxima if $f''(c) < 0$ and a minima if $f''(c) > 0$.

2. Extreme value exists only if the first non-zero derivative is of even order.

Example 7.4. Examine the function $(x - 3)^5 (x + 1)^4$ for extreme values.

Solution. Clearly f is derivable.

Let $$f(x) = (x-3)^5 (x+1)^4$$

$\therefore$ $$f'(x) = (x-3)^4 (x+1)^3 [9x - 7]$$

$$f''(x) = 8(x-3)^3 (x+1)^2 (9x^2 + 14x + 1)$$

$$f'''(x) = 24(x-3)^2 (x+1)(21x^3 - 49x^2 + 7x + 13)$$

$$f''''(x) = 24(x-3)(3x-1)(21x^3 - 49x^2 + 7x + 13) + 168(x+3)^2 (x+1)(9x^2 - 14x + 1)$$

$$f'''''(x) = 48(3x-5)(21x^3 - 49x^2 + 7x + 13) + 336(x-3)(9x^2 - 14x + 1)(3x-1) + 336(x-3)^2 (x+1)(9x-7)$$

Now f' vanishes for $x = -1, 3, \frac{7}{9}$.

Let us now test these for extreme values.

At $x = -1$, f'''' is the first non-vanishing derivative and this is negative. Thus $x = -1$ is a point of maxima.

At $x = 3$ the first non-vanishing derivative is the fifth which is of odd order. Therefore, the function has neither maximum nor minimum at $x = 3$. Such a point is called the *point of inflexion* of the function.

At $x = \frac{7}{9}$, f'' is the first non-vanishing derivative and is positive and therefore, it is a point of minima.

EXERCISE

1. Show that the maximum value of the function $(x - 1)(x - 2)(x - 3)$ is $\frac{2\sqrt{3}}{9}$ at $x = 2 - \frac{1}{\sqrt{3}}$.
2. Show that $x^5 - 5x^4 + 5x^3 - 1$ has a maxima at $x = 1$ and a minima at $x = 3$ and neither at $x = 0$.
3. Find the maximum and the minimum as well as the greatest and the least value of $x^3 - 12x^2 + 45x$ in the interval $[0, 7]$.
4. Find the maximum or minimum if any of
$$\frac{x^4}{(x-1)(x-3)^3}.$$
5. Show that the maximum value of $(1/x)^x$ is $e^{1/e}$.
6. Show that the maximum value of $(\log x)/x$ in $0 < x < \infty$ is $1/e$.
7. Show that $\sin x(1 + \cos x)$ is maximum at $x = \pi/3$.
8. If $(x - a)^{2n}(x - b)^{2m+1}$, where m and n are positive integers, is the derivative of a function f then show that $x = b$ gives a minimum but $x = a$ gives neither a maximum nor a minimum.
9. Show that the semi-vertical angle of a cone of maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.
10. Show that the volume of the greatest cylinder which can be inscribed in a cone of height h and semi-vertical angle α is $(4/27)\pi h^3 \tan^2\alpha$.
11. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius a is $2a/\sqrt{3}$.

ANSWERS

3. Max 54, min 50, greatest 70, least 0.
4. Max at $\frac{6}{5}$, min at $x = 0$.

7.2 INDETERMINATE FORMS

We shall now discuss the evaluation of limits of functions generally known as *Indeterminate forms*. They are not indeterminate but have acquired this name by usage of the word.

In general, the limit of $\phi(x)/\psi(x)$ when $x \to a$, in case the limits of both the functions exist, is equal to the limit of the numerator divided by the limit of the denominator. But what happens when both these limits are zero? The division (0/0) then becomes meaningless. A case like this is known as Indeterminate form. Other such forms are ∞/∞, $0 \times \infty$, $\infty - \infty$, 0^0, 1^∞, and ∞^0. Ordinary methods of evaluating the limits are of little help. Particular methods are required to evaluate these peculiar limits. We shall now discuss these particular methods, generally called **L' Hopital rule**, due to the French mathematician L' Hopital (also called L' Hospital).

It should, however, be clearly understood, that in what follows, we do not find the value of 0/0 or of any of the other indeterminate forms. We only find the limits of combinations of functions which assume these forms when the limits of functions are taken separately.

7.2.1 It will be of help to remember the following points concerning the limits and continuity:

(*i*) $\lim\limits_{x\to a} f(x) = l$

i.e., the function f is defined for every point ξ of the deleted neighbourhood $]a-\delta, a+\delta[$ of a and $f(\xi) \to l$ as $\xi \to a$.

(*ii*) Continuity of $f(x)$ at $x = a \Rightarrow \lim\limits_{x\to a} f(x) = f(a)$,

i.e., the function f is defined for every point ξ of a neighbourhood $]a-\delta, a+\delta[$ of a and $f(\xi) \to f(a)$ as $\xi \to a$.

(*iii*) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = l\ (\neq 0 \text{ or } \infty)$, then

(*a*) $\lim\limits_{x\to a} g(x) \neq 0\ (\text{or } \infty) \Rightarrow \lim\limits_{x\to a} f(x)$ exists finitely

(*b*) $\lim\limits_{x\to a} g(x) = 0\ (\text{or } \infty) \Rightarrow \lim\limits_{x\to a} f(x) = 0\ (\text{or } \infty)$

for

$$\lim f(x) = \lim\left(\frac{f(x)}{g(x)} \cdot g(x)\right) = \lim\left(\frac{f(x)}{g(x)}\right) \lim g(x) = l \cdot \left(\lim g(x)\right)$$

7.2.2 Indeterminate Form, 0/0

We shall now discuss some theorems concerning the indeterminate form 0/0. The reader will do well to note the differences in the hypothesis and the line of proof of the theorems.

Theorem 7.3. *If f, g be two functions such that*

(*i*) $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$, *and*

(*ii*) $f'(a)$, $g'(a)$ *exist and* $g'(a) \neq 0$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Since the functions f and g are derivable at $x = a$, therefore, they are continuous thereat, *i.e.*,

$$\lim_{x\to a} f(x) = f(a) \quad \text{and} \quad \lim_{x\to a} g(x) = g(a)$$

Thus from condition (*i*), $f(a) = 0 = g(a)$.

Also

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{f(x)}{x - a}$$

and

$$g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a} = \lim_{x \to a} \frac{g(x)}{x - a}$$

$$\therefore \qquad \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Note. Condition (*i*) can be replaced by $f(a) = g(a) = 0$.

Theorem 7.4. (*L' Hopital's Rule for* 0/0 *form*). *If f, g be two functions such that*

(*i*) $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$,

(*ii*) $f'(x)$, $g'(x)$ *exist and* $g'(x) \neq 0$, $\forall\ x \in]a - \delta, a + \delta[$, $\delta > 0$ *except possibly at a, and*

(*iii*) $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ *exists*,

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Let us define two functions F and G, such that

$$F(x) = \begin{cases} f(x), \forall\ x \in]a - \delta, a + \delta[\text{ except } a \\ \lim_{x \to a} f(x), \text{ at } x = a \end{cases}$$

$$G(x) = \begin{cases} g(x), \forall\ x \in]a - \delta, a + \delta[\text{ except } a \\ \lim_{x \to a} g(x), \text{ at } x = a \end{cases}$$

Clearly F and G are continuous and derivable on $]a-\delta, a + \delta[$ except possibly at a,

Also, since

$$\lim_{x \to a} F(x) = \lim_{x \to a} f(x) = F(a)$$

and

$$\lim_{x \to a} G(x) = \lim_{x \to a} g(x) = G(a)$$

hence F and G are continuous at a as well.

Let x be a point of $]a-\delta, a + \delta[$ other than a.

For $x > a$, F and G satisfy the conditions of Cauchy's Mean Value Theorem in $[a, x]$ so that

$$\frac{F(x) - F(a)}{G(x) - G(a)} = \frac{F'(\xi)}{G'(\xi)}, \text{ where } a < \xi < x$$

But

$$F(a) = 0 = G(a)$$

$$\therefore \quad \frac{F(x)}{G(x)} = \frac{F'(\xi)}{G'(\xi)} \qquad ...(1)$$

Proceeding to limits,

$$\lim_{x \to a+0} \frac{F(x)}{G(x)} = \lim_{\xi \to a+0} \frac{F'(\xi)}{G'(\xi)} = \lim_{x \to a+0} \frac{F'(x)}{G'(x)}$$

$$\Rightarrow \quad \lim_{x \to a+0} \frac{f(x)}{g(x)} = \lim_{x \to a+0} \frac{f'(x)}{g'(x)}$$

For $x < a$, we can similarly prove that

$$\lim_{x \to a-0} \frac{f(x)}{g(x)} = \lim_{x \to a-0} \frac{f'(x)}{g'(x)}$$

Hence

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

It may be noted in the above theorem that if $g'(x) = 0$ at any point of the interval $]a-\delta, a+\delta[$, then $f'(x)$ also vanishes thereat so that $f'(x)/g'(x)$ then takes up the indeterminate form 0/0. In such situations we have to proceed to the next two theorems which may be considered as generalisations of these theorems and may be proved by repeated applications of the above theorems.

Theorem 7.5. (*Generalised L' Hopital's Rule for* 0/0 *form*). *If f, g be two functions such that*

(*i*) $f^n(x)$, $g^n(x)$ *exist, and* $g'(x) \neq 0$ ($r = 0, 1, 2, ..., n$) *for any x in* $]a-\delta, a+\delta[$ *except possibly at* $x = a$,

(*ii*) *when* $x \to a$, $\begin{cases} lim\ f(x) = lim\ f'(x) = ... = lim\ f^{n-1}(x) = 0 \\ lim\ g(x) = lim\ g'(x) = ... = lim\ g^{n-1}(x) = 0 \end{cases}$

and (*iii*) $\lim\limits_{x \to a} \dfrac{f^n(x)}{g^n(x)}$ *exists*,

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^n(x)}{g^n(x)}.$$

Theorem 7.6. *If f, g be two functions such that*

$$(i)\quad \text{when } x \to a,\ \begin{cases} \lim f(x) = \lim f'(x) = \ldots = \lim f^{n-1}(x) = 0 \\ \lim g(x) = \lim g'(x) = \ldots = \lim g^{n-1}(x) = 0 \end{cases}$$

and (ii) $f^n(a)$, $g^n(a)$ *exist, and* $g^n(a) \neq 0$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f^n(a)}{g^n(a)}.$$

[*Hint*. Using Theorem (7.3) it can be easily seen that

$$\lim_{x \to a} \frac{f^{n-1}(x)}{g^{n-1}(x)} = \frac{f^n(a)}{g^n(a)}$$

Also repeated applications of theorem (7.4) gives

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^{n-1}(x)}{g^{n-1}(x)}\Bigg]$$

Remark. The rule in the form of theorem 7.4 or 7.5 is more useful than the other forms because we may cancel out common factors or affect any other simplification in the quotient $f'(x)/g'(x)$ before proceeding to the limits. Also repeated application of the rule is possible in this form.

L' Hopital's rule holds even in the case when each $f(x)$ and $g(x)$ tends to ∞ when $x \to a$ or when $x \to \infty$. Theorems 7.2 and 7.8 will show how these cases can be handled in an exactly similar fashion.

Theorem 7.7. (*L' Hopital's rule for infinite limits*). *If f, g be two functions such that*

(i) $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = 0.$

(ii) $f'(x)$, $g'(x)$ *exist, and* $g'(x) \neq 0$, $\forall\ x > 0$ *except possibly at* ∞, *and*

(iii) $\lim_{x \to \infty} \frac{f'(x)}{g'(x)}$ *exists*,

then

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}.$$

Let us put $z = 1/x$ so that $z \to 0 + 0$ as $x \to \infty$, and define two functions F and G where $F(z) = f(1/z)$ and $G(z) = g(1/z)$.

We see that the functions F and G satisfy the conditions of Theorem 7.4, viz.,

(i) $\lim_{z \to 0+0} F(z) = \lim_{z \to 0+0} G(z) = 0,$

(ii) $F'(z)$, $G'(z)$ exist and $G'(z) \neq 0$, $\forall\ z$ in $]-\delta, \delta[$ except possibly at $z = 0$, and

(*iii*) $\lim\limits_{z\to 0+0} \dfrac{F'(z)}{G'(z)}$ exists.

Consequently

$$\lim_{z\to 0+0} \frac{F(z)}{G(z)} = \lim_{z\to 0+0} \frac{F'(z)}{G'(z)}$$

$$\Rightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

7.2.3 Indeterminate Form, ∞/∞

If $f(x)$ and $g(x)$ both tend to ∞ as $x \to a$ then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ takes ∞/∞ form. We shall prove L' Hopital's rule for this indeterminate form when x tends to a finite limit a. The rule, however, holds good even for infinite limits, $x \to \infty$ and may be deduced from the following theorem by the procedure indicated in Theorem 7.7.

Theorem 7.8*. (*L' Hopital's rule for* ∞/∞ *form*). *If f, g be two functions such that*

(*i*) $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = \infty$,

(*ii*) $f'(x), g'(x)$ *exist and* $g'(x) \neq 0$, $\forall$ x *in* $]a-\delta, a+\delta[$, $\delta > 0$ *except possibly at a, and*

(*iii*) $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)} = l$,

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = l \cdot$$

Consider the function

$$\phi(x) = f(x) - lg(x) \qquad ...(1)$$

[Division by $g(x)$ would indicate that $f(x)/g(x) \to l$, if $\phi(x)/g(x) \to 0$. With this in mind we proceed to prove that the later limit is actually equal to zero.]

Clearly, in view of condition (*ii*), $\phi(x)$ is derivable and continuous in $]a-\delta, a+\delta[$, except possibly at $x = a$. Also $g'(x) \neq 0$ thereat.

$$\therefore \qquad \frac{\phi'(x)}{g'(x)} = \frac{f'(x)}{g'(x)} - l$$

But condition (*iii*) implies that $\exists$ a positive $\delta_1 < \delta$ and $\varepsilon > 0$, such that

$$\left| \frac{f'(x)}{g'(x)} - l \right| < \frac{\varepsilon}{2}, \text{ when } |x-a| < \delta_1$$

$$\therefore \qquad \left|\frac{\phi'(x)}{g'(x)}\right| < \frac{\varepsilon}{2}, \text{ for } |x-a| < \delta_1 \qquad \ldots(2)$$

Let $x \neq a$ be a point of $]a-\delta_1, a+\delta_1[$.

For $x > a$, $\phi(x)$ and $g(x)$ satisfy all the conditions of Cauchy's Mean Value Theorem in $[x, a+\delta_1]$, so that

$$\frac{\phi(x)-\phi(a+\delta_1)}{g(x)-g(a+\delta_1)} = \frac{\phi'(\xi)}{g'(\xi)}, \text{ when } x < \xi < a+\delta_1 \qquad \ldots(3)$$

Now in view of (2),

$$\left|\frac{\phi'(\xi)}{g'(\xi)}\right| < \frac{\varepsilon}{2}$$

Also since $g(x) \to \infty$ as $x \to a$, we have

(α) $\exists$ a positive $\delta_2 < \delta_1$ such that

and
$$\left.\begin{array}{l} g(x) > 0 \\ g(x) > (2/\varepsilon)\,|\phi(a+\delta_1)| \end{array}\right\}, \text{ when } (x-a) < \delta_2$$

(β) $\exists$ a positive $\delta_3 < \delta_1$ such that

$$g(x) > g(a+\delta_1), \text{ when } (x-a) < \delta_3$$

Thus for $\delta_4 = \min(\delta_2, \delta_3)$, we have

and
$$\left.\begin{array}{l} g(x) > 0 \\ g(x) - g(a+\delta_1) < g(x) \end{array}\right\}, \text{ when } (x-a) < \delta_4$$

Accordingly (3) gives

$$\frac{|\phi(x)-\phi(a+\delta_1)|}{g(x)} < \frac{\varepsilon}{2} \qquad \ldots(4)$$

Now

$$\left|\frac{\phi(x)}{g(x)}\right| \leq \left|\frac{\phi(x)-\phi(a+\delta_1)}{g(x)}\right| + \left|\frac{\phi(a+\delta_1)}{g(x)}\right|$$

$$\leq \left|\frac{\phi(x)-\phi(a+\delta_1)}{g(x)}\right| + \left|\frac{\phi(a+\delta_1)}{g(x)}\right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \text{ for } (x-a) < \delta_4$$

Thus $\lim\limits_{x \to a+0} \dfrac{\phi(x)}{g(x)} = 0$

For $x < a$, we can similarly prove that

$$\lim_{x \to a-0} \frac{\phi(x)}{g(x)} = 0.$$

Hence
$$\lim_{x \to a} \frac{\phi(x)}{g(x)} = 0 \qquad ...(5)$$

But since $\phi(x) = f(x) - lg(x)$, and $g(x) \neq 0$ for any $x \in]a - \delta_4, a + \delta_4[$ by (α), therefore, dividing by $g(x)$ and taking limits,

$$\lim_{x \to a} \left[\frac{f(x)}{g(x)} - l \right] = \lim_{x \to a} \frac{\phi(x)}{g(x)} = 0$$

so that $\lim_{x \to a} \frac{f(x)}{g(x)} = l$

Particular case: $l = 0$, *i.e.*, $\lim_{x \to a} \frac{f'(x)}{g'(x)} = 0.$

The auxiliary function $\phi(x)$ reduces to $f(x)$, so that we have to proceed without introducing any such function. And we shall have the simplification, $\lim_{x \to a} \frac{f'(x)}{g'(x)} = 0$, in place of $\lim_{x \to a} \frac{\phi'(x)}{g'(x)} = 0.$

So proceeding in a similar way we may prove the *theorem*:

If f, g be two functions such that

(*i*) $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty.$

(*ii*) $f'(x), g'(x)$ *exist and* $g'(x) \neq 0$, *in a deleted neighbourhood of a, and*

(*iii*) $\lim_{x \to a} \frac{f'(x)}{g'(x)} = 0$

then
$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0.$$

Remarks 1. $\lim_{x \to a} \frac{f(x)}{g(x)}$, when $f(x)$ and $g(x)$ both tend to infinity, can be dealt with in the same way as 0/0 form.

2. The above theorem holds good in the case of infinite limits $x \to \infty$ as well.

3. Sometimes repeated applications of L'Hopital's rule may be necessary to evaluate a limit. We must then ensure at each step that the expression to which the rule is applied, is actually an indeterminate form.

4. The forms 0/0 and ∞/∞ can be interchanged and so care should be taken to select the form which would enable us to evaluate the limit most quickly.

7.2.4 (a) Form $0 \times \infty$

When $f(x) \to 0$ and $g(x) \to \infty$ as $x \to a$, $f(x) \cdot g(x)$ takes $0 \times \infty$ form.

However $f(x) \cdot g(x)$ may be expressed as

$$\frac{f(x)}{1/g(x)} \text{ or } \frac{g(x)}{1/f(x)}$$

which has respectively 0/0 and ∞/∞ forms.

(*b*) Form $\infty - \infty^0$

This can be reduced to the form 0/0 or ∞/∞. For

$$f(x) - g(x) = \frac{\dfrac{1}{g(x)} - \dfrac{1}{f(x)}}{\dfrac{1}{f(x)\, g(x)}}, \qquad \left(\frac{0}{0} \text{ form}\right)$$

(*c*) Form 0°, ∞^0

These forms can be made to depend upon one of the previous forms by putting $k = \{f(x)\}^{g(x)}$, so that

$$\log k = g(x) \cdot \log f(x)$$

$\therefore$ $$\lim \log k = \lim \{g(x) \log f(x)\}$$

Also $$\lim k = \lim e^{\log k} = e^{\lim \log k}$$

Thus the limit may be evaluated by one of the previous methods.

7.2.5 Let us now evaluate some limits which take up these forms, we shall not hesitate to make use of certain known limits, such as $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$, $\lim\limits_{x\to 0} \dfrac{\tan x}{x}$, $\lim\limits_{x\to\infty} \left(1 + \dfrac{1}{x}\right)^x$, etc. or expansions of functions such as $\log(1 + x)$, $\sin x$, etc. either in the beginning or at some intermediate stage because it simplifies and shortens the process of evaluation of a limit to a considerable extent.

Example 7.5. Evaluate $\lim\limits_{x\to 0} \dfrac{x - \tan x}{x^3}$.

Solution. Let $\dfrac{x - \tan x}{x^3} = \dfrac{f(x)}{g(x)}$, where $f(x) = x - \tan x$ and $g(x) = x^3$.

Now

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} (x - \tan x) = 0$$

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} x^3 = 0$$

$\therefore$ $\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)}$ is of $\dfrac{0}{0}$ form, so that L'Hopital's rule is applicable, *i.e.*,

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{f'(x)}{g'(x)}$$

Hence
$$\lim_{x\to 0}\frac{x-\tan x}{x^3}=\lim_{x\to 0}\frac{1-\sec^2 x}{3x^2}$$

$$=-\frac{1}{3}\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^2=-\frac{1}{3}$$

Example 7.6. Evaluate $\lim_{x\to 0}\dfrac{xe^x-\log(1+x)}{x^2}$.

Solution. It is a 0/0 form and therefore,

$$\lim_{x\to 0}\frac{xe^x-\log(1+x)}{x^2}=\lim_{x\to 0}\frac{xe^x+e^x-\dfrac{1}{1+x}}{2x}, \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0}\frac{xe^x+2e^x+\dfrac{1}{(1+x)^2}}{2}=\frac{3}{2}.$$

Example 7.7. Find $\lim_{x\to 0}\dfrac{(1+x)^{1/x}-e}{x}$.

Solution. Since $\lim_{x\to 0}\dfrac{(1+x)^{1/x}-e}{x}$, therefore it is a 0/0 form

$$\therefore \quad \lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x}=\lim_{x\to 0}\frac{\dfrac{d}{dx}(1+x)^{1/x}}{1}$$

$$=\lim_{x\to 0}\frac{(1+x)^{1/x}\{x-(1+x)\log(1+x)\}}{x^2(1+x)}$$

$$=e\cdot\lim_{x\to 0}\frac{x-(1+x)\log(1+x)}{x^2(1+x)} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$=e\cdot\lim_{x\to 0}\frac{-\log(1+x)}{2x+3x^2} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$=e\cdot\lim_{x\to 0}\frac{-1}{(2+6x)(1+x)}=-\frac{e}{2}$$

Aliter. Let us first find an algebraic expansion for $(1+x)^{1/x}$.

Let $y=(1+x)^{1/x}$

$$\log\ y = \frac{1}{x}\log\left(1+x\right)$$

$$= \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right), \text{ if } |x| < 1$$

$$= 1 - \frac{x}{2} + \frac{x^2}{3} - \ldots$$

$$\therefore \qquad y = \exp\left(1 - \frac{x}{2} + \frac{x^2}{3} - \ldots\right)$$

$$= e\cdot\exp\left(-\frac{x}{2} + \frac{x^2}{3} - \ldots\right)$$

$$= e\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \ldots\right) + \frac{1}{2!}\left(-\frac{x}{2} + \frac{x^2}{3} - \ldots\right)^2 + \ldots\right]$$

$$= e\left[1 - \frac{x}{2} + \frac{11}{24}x^2 - \ldots\right]$$

$$\therefore \qquad \lim \frac{\left(1+x\right)^{1/x} - e}{x} = \lim_{x\to 0} \frac{e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 - \ldots\right) - e}{x} = -\frac{e}{2}$$

Example 7.8. Find $\lim\limits_{x\to 1-0} \dfrac{\log\left(1-x\right)}{\cot\left(\pi x\right)}$.

Solution. It is a ∞/∞ form and therefore,

$$\lim_{x\to 1-0} \frac{\log\left(1-x\right)}{\cot\left(\pi x\right)} = \lim_{x\to 1-0} \frac{\dfrac{-1}{1-x}}{-\pi\ \text{cosec}^2\left(\pi x\right)}$$

$$= \lim_{x\to 1-0} \frac{\sin^2\left(\pi x\right)}{\pi\left(1-x\right)} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 1-0} \frac{\pi\sin\left(2\pi x\right)}{-\pi} = 0\ .$$

Example 7.9. Evaluate $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2} - \dfrac{1}{\sin^2 x}\right)$.

Solution. It is a $(\infty - \infty)$ form, we therefore write

$$\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$$

$$=\lim_{x\to 0}\frac{\sin^2 x-x^2}{x^2\sin^2 x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0}\frac{\sin 2x-2x}{2x\sin^2 x+x^2\sin 2x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0}\frac{\cos 2x-1}{\left(\sin^2 x+2x\sin 2x+x^2\cos 2x\right)} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0}\frac{-2\sin 2x}{3\sin 2x+6x\cos 2x-2x^2\sin 2x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0}\frac{-\cos 2x}{3\cos 2x-4x\sin 4x-x^2\cos 2x}=-\frac{1}{3}$$

Aliter.

$$\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)=\lim_{x\to 0}\frac{\sin^2 x-x^2}{x^2\sin^2 x}$$

$$=\lim_{x\to 0}\frac{\sin^2 x-x^2}{x^4}\left(\frac{x}{\sin x}\right)^2$$

$$\left[\text{Using }\lim_{x\to 0}\left(\frac{x}{\sin x}\right)=1\right]$$

$$=\lim_{x\to 0}\frac{\sin^2 x-x^2}{x^4} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0}\frac{\sin 2x-2x}{4x^3} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0}\frac{\cos 2x-1}{6x^2}$$

$$=\lim_{x\to 0}\left[-\frac{1}{3}\left(\frac{\sin x}{x}\right)^2\right]=-\frac{1}{3}$$

Note. It should be noted that when a given function can be put as a product of two or more factors, the limit of each of which can be easily found, then limit of the entire function can be determined by evaluating the limit of each factor separately, provided that the product of these limits is not an indeterminate form.

Example 7.10. Evaluate $\lim\limits_{x\to 0+0} (\sin x \log x)$.

Solution. It is a $(0\times\infty)$ form. Let us write

$$\lim_{x\to 0+0} (\sin x \log x) = \lim_{x\to 0+0} \frac{\log x}{\operatorname{cosec} x} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= -\lim_{x\to 0+0} \frac{1/x}{\operatorname{cosec} x \cot x} = -\lim_{x\to 0+0} \left(\frac{\sin x}{x}\right) \tan x = 0$$

Example 7.11. Evaluate $\lim\limits_{x\to 0} \left(\dfrac{\tan x}{x}\right)^{1/x^2}$.

Solution. It is a (1^{∞}) form. Therefore, let $K = \left(\dfrac{\tan x}{x}\right)^{1/x^2}$, so that

$$\log K = \frac{1}{x^2} \log \left(\frac{\tan x}{x}\right)$$

$$\therefore \quad \lim_{x\to 0} \log K = \lim_{x\to 0} \frac{\log \left(\dfrac{\tan x}{x}\right)}{x^2} = \lim_{x\to 0} \frac{\dfrac{\sec^2 x}{\tan x} - \dfrac{1}{x}}{2x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{x \sec^2 x - \tan x}{2x^2 \tan x} = \lim_{x\to 0} \frac{\sec^2 x \tan x}{2\tan x + x \sec^2 x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\tan x}{\sin 2x + x} = \lim_{x\to 0} \frac{\sec^2 x}{2\cos 2x + 1} = \frac{1}{3} \qquad \left(\frac{0}{0} \text{ form}\right)$$

i.e., $$\lim_{x\to 0} \log K = \frac{1}{3} \Rightarrow \lim_{x\to 0} K = e^{1/3}$$

$$\therefore \quad \lim_{x\to 0} K = \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x^2} = e^{1/3}$$

Example 7.12. Evaluate $\lim\limits_{x\to 1-0} (1 - x^2)^{1/(\log(1-x))}$.

Solution. It is a (0^0) form. Therefore, let $K = (1 - x^2)^{1/(\log(1-x))}$, so that

$$\log K = \frac{\log\left(1 - x^2\right)}{\log\left(1 - x\right)}$$

$$\therefore \quad \lim_{x \to 1-0} \log K = \lim_{x \to 1-0} \frac{\log\left(1 - x^2\right)}{\log\left(1 - x\right)} = \lim_{x \to 1-0} \frac{2x(1 - x)}{1 - x^2} = \lim_{x \to 1-0} \frac{2x}{1 + x} = 1 \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$\Rightarrow \lim_{x \to 1-0} \log K = 1 \Rightarrow \lim_{x \to 1-0} K = e.$$

EXERCISE

Evaluate the following limits:

1. $\lim\limits_{x \to 0} \dfrac{x - \sin x}{x^3}$
2. $\lim\limits_{x \to 0} \dfrac{x - \log\left(1 + x\right)}{1 - \cos x}$
3. $\lim\limits_{x \to 0} \dfrac{\log\left(1 + x^3\right)}{\sin^3 x}$
4. $\lim\limits_{x \to \infty} \dfrac{x^3}{e^x}$
5. $\lim\limits_{x \to \pi/2+0} \dfrac{\log\left(x - \dfrac{\pi}{2}\right)}{\tan x}$
6. $\lim\limits_{x \to \infty} x \tan \dfrac{1}{x}$
7. $\lim\limits_{x \to 0} \left(\cot^2 x - \dfrac{1}{x^2}\right)$
8. $\lim\limits_{x \to \pi/2} (\sec x - \tan x)$
9. $\lim\limits_{x \to 0+0} (\cot x)^{\sin x}$
10. $\lim\limits_{x \to \pi/2} (\sin x)^{\tan x}$
11. $\lim\limits_{x \to 0} \dfrac{\left(1 + x\right)^{1/x} - e + \dfrac{1}{2}ex}{x^2}$

    [*Hint*. Use algebraic expansion of $(1 + x)^{1/x}$.]
12. $\lim\limits_{x \to 0} \dfrac{e^x - 2 \cos x + e^{-x}}{x \sin x}$
13. $\lim\limits_{x \to 0} \left(1 - x\right)^{1/x}$
14. $\lim\limits_{x \to a} \left(2 - \dfrac{x}{a}\right)^{\tan (\pi x/2a)}$
15. Find the values of a and b in order that $\lim\limits_{x \to 0} \dfrac{x(1 + a \cos x) - b \sin x}{x^3}$ may be equal to 1.

ANSWERS

1. $\dfrac{1}{6}$
2. 1
3. 1
4. 0
5. 0
6. 1
7. $-\dfrac{2}{3}$
8. 0
9. 1
10. 1
11. $\dfrac{11e}{24}$
12. 2
13. e^{-1}
14. $e^{2/\pi}$
15. $a = -5/2, b = -3/2$

CHAPTER 8

Elementary Functions

8.1 INTRODUCTION

In this chapter, we propose to introduce the so-called Elementary functions.

$$e^x, \log x, a^x, \sin x, \cos x.$$

The reader is already familiar with these functions but this acquaintance is based on a treatment which was essentially based on intuitive and less rigorous geometrical considerations. Even the question of existence was ignored.

We shall base the study of these functions on the set of real numbers as a complete ordered field, the notion of limit and the convergence of series. Starting from the definitions of these functions, their basic properties will be studied. No attempt, however, will be made to make the discussion exhaustive.

We shall consciously accept an abuse of language in as much as notation for a function will not be distinguished from that for the functional value. Thus instead of denoting the function by cos we shall denote the same as cos x and so on.

As the definitions of function will be based on *Power series*, we start our discussion with a brief (very brief!) study of power series.

8.2 POWER SERIES

The series of the form

$$a_0 + a_1x + a_2x^2 + ... + a_nx^n + ... = \sum_{n=0}^{\infty} a_nx^n \text{ or simply } \Sigma a_nx^n$$

are called *power series* (in x) and the numbers $a_0, a_1, a_2 ... a_n$ are their *coefficients*.

For such series we are not concerned simply with the alternatives 'convergent' and 'divergent', but the more precise question: For what values of x is the series convergent and for what values divergent?

8.2.1 Some simple examples have already come before us:

1. The geometric series Σx^n is convergent for $|x| < 1$, divergent for $|x| \geq 1$.

   For $|x| < 1$, indeed, we have absolute convergence.

2. $\Sigma \frac{x^n}{n!}$ is (absolutely) convergent for every real x; likewise the series $\Sigma(-1)^n \frac{x^{2n}}{2n!}$ and $\Sigma(-1)^n \frac{x^{2n+1}}{(2n+1)!}$.

3. $\Sigma \frac{x^n}{n}$.

   For $x = 1$, diverges (reduces to divergent harmonic series)

   $x = -1$, converges

   $|x| < 1$ absolutely convergent ($\leq |x|^n$, comparison test)

   $|x| > 1$ diverges

4. $\Sigma n^n x^n$ converges for $x = 0$, but diverges for $x \neq 0$.

8.2.2 For $x = 0$, obviously every power series $\Sigma a_n x^n$ is convergent, whatever be the value of the coefficient a_n. The general case is evidently that in which the power series converges for some values of x and diverges for others, while in special instances, the two extreme cases may occur, in which the series converges for every x or for none $x \neq 0$.

In the first of these special cases, when the series converges for every x, we say that the power series is **everywhere convergent**, while in the second (leaving out of account the self-evident point of convergence $x = 0$) when the series converges for no value of x we say that it is **nowhere convergent**. The totality of points x for which the series converges is called its *region of convergence*.

For a power series $\Sigma a_n x^n$, which does not merely converge everywhere or nowhere, a definite positive number R exists such that the series converges for every $|x| < R$ (indeed absolutely), but diverges for every $|x| > R$. The number R is called the **Radius of convergence**, and the interval $]-R, R[$, the *Interval of convergence*, of the given series. The behaviour of the series is much more varied at $x = \pm R$ and is beyond the scope of the present discussion.

8.2.3 For the power series $\Sigma a_n x^n$, put

$$\lim_{n \to \infty} \sup \sqrt[n]{|a_n|} = \alpha \quad \text{and} \quad R = \frac{1}{\alpha}$$

(where $R = +\infty$, for $\alpha = 0$ and $R = 0$, for $\alpha = \infty$.)

By Cauchy's root test, it follows that if

(*i*) $R = 0$, the series is nowhere convergent;

(*ii*) $R = \infty$, the series is everywhere convergent;

(*iii*) $0 < R < \infty$, the series converges absolutely for $|x| < R$, and diverges for $|x| > R$, *i.e.*, R is the radius of convergence.

Exercise. $1 + x + x^2 + \ldots$ converges for $|x| < 1$ and equals $(1 - x)^{-1}$.

8.3 EXPONENTIAL FUNCTIONS

The power series

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ... \qquad ...(1)$$

is everywhere convergent for real x. We proceed now to examine in detail the function represented by this series.

8.3.1 Definition

The function represented by the power series (1) is called the *Exponential function*, denoted, provisionally, by $E(x)$. Thus

$$E(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + ... \qquad ...(2)$$

$$\therefore \qquad E(0) = 1$$

and

$$E(1) = 1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!} + ... \qquad ...(3)$$

The series on the right hand side of (3) converges to a number which lies between 2 and 3. This number is denoted by e, the *Exponential base* and is the same number as represented by

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Thus $E(1) = e$.

8.3.2 The Additional Formula

The function $E(x)$, defined by (2) is continuous and differentiable any number of times, for every x.

By differentiation, we get

$$E'(x) = E(x)$$
$$E''(x) = E(x)$$
$$\vdots$$
$$E^{(n)}(x) = E(x)$$

Further we state (justification may be seen expanding by Taylor's Theorem)* that

$$E(x_1 + x_2) = E(x_1).E(x_2)$$

---

*$E(x) = E(x_1) + \frac{E(x_1)}{1!}(x - x_1) + ...$, for all values of x and x_1.

Replacing x by $x_1 + x_2$, we get

$$E(x_1 + x_2) = E(x_1)\left\{1 + \frac{x_2}{1!} + \frac{x_2^2}{2!} + ...\right\} = E(x_1) \cdot E(x_2)$$

This formula is called the *Addition formula* for the exponential function. It gives further

$$E(x_1 + x_2 + x_3) = E(x_1 + x_2) \cdot E(x_3)$$
$$= E(x_1) \cdot E(x_2) \cdot E(x_3)$$

and repetition of the process gives, for any positive integer q,

$$E(x_1 + x_2 + ... + x_q) = E(x_1) \cdot E(x_2) ... E(x_q) \quad ...(4)$$

If $x_1 = x_2 = x_3 = ... = x_q = x$, we get

$$E(qx) = \{E(x)\}^q \quad ...(5)$$

Hence for $x = 1$,

$$E(q) = \{E(1)\}^q = e^q, \text{ for any positive integer } q$$

But since $E(0) = 1$, therefore the above relation holds for $q = 0$ also.

Hence $E(q) = e^q$ holds for all integers ≥ 0.

Again replacing each x by p/q in (5), we get

$$E\left(q\frac{p}{q}\right) = \left\{E\left(\frac{p}{q}\right)\right\}^q \text{ for positive integers, } p, q$$

or

$$E(p/q) = \{E(p)\}^{1/q} = e^{p/q} \qquad \left[\because\ E(p) = e^p\right]$$

Hence $E(m) = e^m$, for all rational numbers $m \geq 0$.

For any positive *irrational* number ξ there always exists a sequence $\{x_n\}$ of positive rational terms, converging to ξ.

Now for each n

$$E(x_n) = e^x n.$$

When $n \to +\infty$, the left hand side tends to $E(\xi)$, and the right hand side to e^{ξ}, so that we get

$$E(\xi) = e^{\xi}$$

$$\therefore \qquad E(x) = e^x, \text{ for real } x \geq 0 \quad ...(6)$$

Again by Addition formula,

$$E(x) \cdot E(-x) = E(x - x) = E(0) = 1 \quad ...(7)$$

Thus we conclude that $E(x) \neq 0$, for any real x, and that for $x \geq 0$,

$$E(-x) = \frac{1}{E(x)} = \frac{1}{e^x} = e^{-x},$$

Consequently, $E(x) = e^x$ holds for all real x.

8.3.3 Monotonicity

By definition

$$E(x) > 0, \ \forall\ x > 0$$

so that from (7) it follows that

$$E(-x) > 0, \ \forall\ x > 0$$

Hence $E(x) > 0$, for all real x.

Again by definition, for real x,

$$E(x) \to +\infty, \text{ as } x \to +\infty$$

Hence (7) shows that

$$E(x) \to 0 \text{ as } x \to -\infty$$

Also by definition,

$$0 < x_1 < x_2 \Rightarrow E(x_1) < E(x_2)$$

Also it follows from (7) that

$$E(-x_2) < E(-x_1), \text{ when } -x_2 < -x_1 < 0$$

Hence the function E is strictly increasing from 0 to $+\infty$ on the whole real line.

Note. By definition $e^x > \dfrac{x^{n+1}}{(n+1)!}$, for $x > 0$, so that $x^n e^{-x} < \dfrac{(n+1)!}{x}$.

$\therefore \lim\limits_{x \to +\infty} x^n e^{-x} = 0$, for all n

This fact we express by saying that e^x *tends to* $+\infty$ "*faster*" *than any power* of x, as $x \to +\infty$.

8.4 LOGARITHMIC FUNCTIONS (*base e*)

Since the exponential function E is strictly increasing on the set $\mathbf{R}$ of real numbers (*i.e.*, E: $\mathbf{R} \to \mathbf{R}^+$ is one-one, onto), it has inverse function L (or $\log_e$) which is also strictly increasing and whose domain of definition is $\mathbf{R}^+ \ (\equiv E(\mathbf{R}))$, the set of positive reals. Thus L is defined by

or
$$\left.\begin{aligned} E\{L(y)\} &= y, \ (y > 0) \\ L\{E(x)\} &= x, \ (x \text{ real}) \end{aligned}\right\} \quad ...(i)$$

or equivalently, for any real x,

or
$$\left.\begin{aligned} E(x) = y &\Rightarrow L(y) = x \\ e^x = y &\Rightarrow \log_e y = x \end{aligned}\right\} \quad ...(ii)$$

Thus the logarithmic function L (or $\log_e$) is defined for positive values only of the variable.

By definition,

$$\left.\begin{aligned} E(-x) = \frac{1}{y} \Rightarrow L\left(\frac{1}{y}\right) &= -x = -L(y) \\ E(0) = 1 \Rightarrow L(1) &= 0 = \log_e 1 \\ E(1) = e \Rightarrow L(e) &= 1 = \log_e e \end{aligned}\right\} \quad ...(iii)$$

Again

$\because$ $$E(x) \to +\infty \text{ as } x \to +\infty$$

and

$$E(x) \to 0 \text{ as } x \to -\infty$$

$\therefore$ $$L(x) \to +\infty \text{ as } x \to +\infty$$

$$L(x) \to -\infty \text{ as } x \to 0$$

Writing $u = E(x_1)$, $v = E(x_2)$ or $L(u) = x_1$, $L(v) = x_2$ in (4), we get

$$E(x_1 + x_2) = uv$$

$$\Rightarrow L(uv) = x_1 + x_2 = L(u) + L(v)$$

which is a familiar property of the logarithmic function and which makes logarithms a useful tool for computation.

Since the function E is differentiable, therefore, its inverse function L is also differentiable.

Hence differentiating (*i*), we get

$$L'\{E(x)\} \cdot E(x) = 1$$

Writing $E(x) = y$, we get

$$L'(y) = \frac{1}{y} \qquad ...(iv)$$

which implies that

$$L(y) = \int_1^y \frac{dx}{x} \qquad ...(8)$$

Quite often (8) is taken as the definition of the logarithmic function and thus the starting point of the theory of the logarithmic and the exponential functions.

Note. In theoretical investigations, it is always more convenient to use the so-called *natural logarithms*, that is to say, those with the base e. Hence in our further discussion, log x *shall always stand for* $L(x)$ or $\log_e x$.

8.4.1 Generalised Power Functions

The meaning of a^x is well understood when a is any positive real number and x is any rational number. We shall now give a meaning to a^x when x is any real number whatsoever. We define thus:

Definition. $a^x = E(x \log a)$, for all x and $a > 0$.

Evidently the range of a^x is the set $\mathbf{R}^+$ of positive reals, *i.e.*,

$$a^x > 0, \quad \forall\ x \qquad ...(9)$$

$\therefore$ $$a^x \cdot a^y = E(x \log a) \cdot E(y \log a)$$

$$= E\{(x + y) \log a\} = a^{x+y}$$

Thus $$a^x \cdot a^y = a^{x+y} \qquad ...(10)$$

Let us now verify that this definition of a^x is consistent with that already known to us for x, an integer or a rational number.

(*i*) Let $x = n$, a positive integer.

$$\therefore \quad a^n = E(n \log a) = E[\log a + \log a + \ldots n \text{ times}]$$
$$= E(\log a) \cdot E(\log a) \ldots n \text{ times}$$
$$= a \cdot a \ldots n \text{ times}$$

(*ii*) Now let $x = -n$, n being a positive integer.

$$\therefore \quad a^{-n} = E(-n \log a)$$
$$= E[(-\log a) + (-\log a) + \ldots n \text{ times}]$$
$$= E\left[\log \frac{1}{a} + \log \frac{1}{a} + \ldots n \text{ times}\right]$$
$$= E\left(\log \frac{1}{a}\right) \cdot E\left(\log \frac{1}{a}\right) \ldots n \text{ times}$$
$$= \frac{1}{a} \cdot \frac{1}{a} \ldots n \text{ times}$$

Thus, $E(x \log a)$ has the same meaning as a^x when x is an integer.

(*iii*) Let now $x = p/q$, where p, q are integers, and q is positive.

Now

$$E\left(\frac{p}{q} \log a\right) = a^{p/q}$$

$$\therefore \quad \left[E\left(\frac{p}{q} \log a\right)\right]^q = a^p = E(p \log a)$$

so that $E\left(\frac{p}{q} \log a\right)$ is qth root of $E(p \log a)$.

Thus, $a^{p/q}$ is a qth root of a^p.

Hence, the definition holds good when x is a rational number.

Thus, the above definition of a^x agrees with what is already known to us about a^x.

8.4.2 Logarithmic Functions (*any base*)

Definition. $a^x = y \Leftrightarrow \log_a y = x$.

Since y is always positive, therefore the logarithmic function, $\log_a$, is defined for positive values only of the variable.

Evidently $$a^{-x} = \frac{1}{y}$$

$$\therefore \quad \log_a \frac{1}{y} = -x = -\log_a y$$

Also, from definition,

$$\log_a 1 = 0, \log_a a = 1$$

It may be easily shown that

$$\log_a x + \log_a y = \log_a (xy)$$
$$\log_a x - \log_a y = \log_a (x/y)$$
$$\log_a x^y = y \log_a x$$
$$\log_b x \cdot \log_a b = \log_a x$$
$$\log_b a \cdot \log_a b = 1$$

Exercise. Show that $\lim_{x\to\infty} \frac{\log x}{x^a} = 0, \ a > 0.$

[*Hint*: Use $\lim_{n\to\infty} x^n/e^x = 0.$]

8.5 TRIGONOMETRIC FUNCTIONS

We are now in a position to introduce rigorously the circular functions, employing purely the arithmetical methods. For this purpose, we consider the power series, everywhere convergent (absolutely and uniformly) and the functions represented by them.

Definition.

$$C(x) = 1 - \frac{x^2}{2!} + \frac{x^2}{2!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots \ \forall\ x$$

$$S(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots \ \forall\ x$$

Each of these series represents a function everywhere continuous and differentiable any number of times in succession. The properties of these functions will be established, taking as starting point their expansions in series form, and it will be seen finally that these coincide with the functions $\cos x$ and $\sin x$ with which we are familiar from elementary studies, *i.e.*, $C(x) \equiv \cos x$ and $S(x) \equiv \sin x$.

8.5.1 Properties of the Functions ($C(x)$, $S(x)$)

(*i*) The functions $C(x)$ and $S(x)$ are continuous and derivable for all x; in fact it may easily be seen that

$$C'(x) = -S(x) \text{ and } S'(x) = C(x)$$

Also

$$C^{iv}(x) = C(x) \text{ and } S^{iv}(x) = S(x)$$

(*ii*) From definitions,

$$S(0) = 0, \ C(0) = 1$$

$$S(-x) = -x - \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} - \ldots + (-1)^n \frac{(-x)^{2n+1}}{(2n+1)!} + \ldots$$

$$= -\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + ... \right] = -S(x) \quad \forall\ x$$

Similarly, $C(-x) = C(x) \quad \forall\ x$.

(*iii*) **The Addition Theorems.** These functions, like the exponential function, satisfy simple addition theorems, by means of which they can then be further examined.

First Method. By Taylor's expansion for any two variables, x_1 and x_2 (since the two series converge everywhere absolutely).

$$C(x_1 + x_2) = C(x_1) + \frac{C'(x_1)}{1!}x_2 + \frac{C''(x_1)}{2!}x_2^2 + ...$$

$$= C(x_1) - \frac{S(x_1)}{1!}x_2 - \frac{C(x_1)}{2!}x_2^2 + \frac{S(x_1)}{3!}x_2^3 + ...$$

As this series is absolutely convergent, we may rearrange it in any way we please.

$$\therefore\ C(x_1 + x_2) = C(x_1)\left\{1 - \frac{x_2^2}{2!} + \frac{x_2^4}{4!} - ...\right\} - S(x_1)\left\{x_2 - \frac{x_2^3}{3!} + \frac{x_2^5}{5!} - ...\right\}$$

$$= C(x_1) \cdot C(x_2) - S(x_1) \cdot S(x_2) \qquad ...(11)$$

Similarly,

$$S(x_1 + x_2) = S(x_1) \cdot C(x_2) + C(x_1) \cdot S(x_2) \qquad ...(12)$$

Second Method. For any fixed value of x_2, consider the functions

$$f(x_1) = S(x_1 + x_2) - S(x_1) \cdot C(x_2) - C(x_1) \cdot S(x_2)$$

$$g(x_1) = C(x_1 + x_2) - C(x_1) \cdot C(x_2) + S(x_1) \cdot S(x_2)$$

Differentiating with respect to x_1, we get

$$f'(x_1) = C(x_1 + x_2) - C(x_1) \cdot C(x_2) + S(x_1) \cdot S(x_2) = g(x_1)$$

$$g'(x_1) = -S(x_1 + x_2) + S(x_1) \cdot C(x_2) + C(x_1) \cdot S(x_2) = -f(x_1)$$

$$\therefore \qquad \frac{d}{dx_1}\left[f^2(x_1) + g^2(x_1) \right] = 2f(x_1)f'(x_1) + 2g(x_1)g'(x_1)$$

$$= 2f(x_1)\, g(x_1) - 2g(x_1)f(x_1) = 0, \quad \forall\ x_1$$

$\Rightarrow f^2(x_1) + g^2(x_1)$ is a constant, $\forall\ x_1$

Hence for all x_1

$$f^2(x_1) + g^2(x_1) = f^2(0) + g^2(0) = 0$$

$$\Rightarrow \qquad f(x_1) = 0,\ g(x_1) = 0$$

$$\therefore \qquad C(x_1 + x_2) = C(x_1) \cdot C(x_2) - S(x_1) \cdot S(x_2)$$

and

$$S(x_1 + x_2) = S(x_1) \cdot C(x_2) + C(x_1) \cdot S(x_2)$$

The form of these theorems coincides with that of the addition theorems for the functions cosine and sine, with which we are clearly acquainted from an elementary standpoint. With the help of these theorems, we shall now show that the functions C and S satisfy all the other so called purely trigonometrical formulae—in fact C and S are same as the functions cosine and sine. We note, in particular:

(a) Changing x_2 to $-x_2$,

$$C(x_1 - x_2) = C(x_1) \cdot C(x_2) + S(x_1) \cdot S(x_2)$$
$$S(x_1 - x_2) = S(x_1) \cdot C(x_2) - C(x_1) \cdot S(x_2)$$

(b) Writing $x_2 = -x_1$, we deduce that

$$C^2(x_1) + S^2(x_1) = 1 \text{ or } C^2(x) + S^2(x) = 1, \ \forall\ x$$
$$\Rightarrow\ |S(x)| \le 1, |C(x)| \le 1, \ \forall\ x$$

(c) Replacing x_1 and x_2 by x,

$$C(2x) = C^2(x) - S^2(x)$$
$$S(2x) = 2S(x) \cdot C(x)$$

8.5.2 The Number π. The Smallest Positive Root of the Equation $C(x) = 0$

Theorem 8.1. *To prove that there exists a positive number* π, *such that*

$$C(\pi/2) = 0 \text{ and } C(x) > 0, \text{ for } 0 \le x < \pi/2 \qquad ...(13)$$

Consider the interval [0, 2].

We know $C(0) = 1 > 0$; we shall now show that $C(2) < 0$.

Now

$$C(2) = 1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \frac{2^6}{6!} + ...$$
$$= 1 - \frac{2^2}{2!}\left(1 - \frac{2^2}{3.4}\right) - \frac{2^6}{6!}\left(1 - \frac{2^2}{7.8}\right) - ...$$

Since the brackets are all positive, we have

$$C(2) < 1 - \frac{2^2}{2!}\left(1 - \frac{2^2}{3.4}\right) = -\frac{1}{3}$$

so that $C(2)$ is negative.

Thus, the continuous function $C(x)$ is positive at 0 and negative at 2.

$\therefore$ $C(x)$ vanishes at least once between 0 and 2 (by the Intermediate-value theorem). Further, since $S(x)$ is positive in [0, 2], where

$$S(x) = x\left(1 - \frac{x^2}{2.3}\right) + \frac{x^5}{5!}\left(1 - \frac{x^2}{6.7}\right) + ...$$

therefore, the derivative ($-S(x)$) of $C(x)$ is always negative for all values of x between 0 and 2. Consequently $C(x)$ is a (strictly) monotonic decreasing function in [0, 2], and can therefore, vanish at only one point in [0, 2].

Thus, there exists one and only one root of the equation $C(x) = 0$ lying between 0 and 2. Denoting this root by $\pi/2$, we see that $\pi/2$ is the least positive root of the equation $C(x) = 0$.

Clearly $C(x) > 0$, when $0 \le x < \pi/2$.

Using the above results, we deduce that

(*a*) $S(x) > 0$, when $0 < x \le \pi/2$.

Since the derivative of $S(x)$ is non-negative in $[0, \pi/2]$, therefore, $S(x)$ is a strictly monotonic increasing function. Also since $S(0) = 0$, therefore, $S(x)$ is positive for $0 < x \le \pi/2$.

(*b*) As $C^2(\pi/2) + S^2(\pi/2) = 1$ and $C(\pi/2) = 0$,

$$\therefore \qquad S^2(\pi/2) = 1 \Rightarrow S(\pi/2) = \pm 1$$

But, by Lagrange's Mean Value Theorem,

$$S(\pi/2) - S(0) = (\pi/2)C(\alpha) > 0, \text{ where } 0 < \alpha < \pi/2$$

$$\Rightarrow \qquad S(\pi/2) = 1$$

(*c*) $C(\pi) = 2C^2(\pi/2) - 1 = -1$

$$S(\pi) = 2S(\pi/2)\, C(\pi/2) = 0$$

(*d*) $C(2\pi) = 1,\ S(2\pi) = 0.$

(*e*) $C(\pi/2) = 2C^2(\pi/4) - 1.$

$$\therefore \qquad C(\pi/4) = 1/\sqrt{2}$$

rejecting the negative sign, as $C(\pi/4)$ is positive.

Similarly, $S(\pi/4) = 1/\sqrt{2}$

(*f*) It finally follows from the addition theorems that for all x,

$$S\left(\frac{1}{2}\pi - x\right) = C(x), \qquad C\left(\frac{1}{2}\pi - x\right) = S(x)$$

$$S\left(\frac{1}{2}\pi + x\right) = C(x), \qquad C\left(\frac{1}{2}\pi + x\right) = -S(x)$$

$$S(\pi + x) = -S(x), \qquad C(\pi + x) = -C(x)$$

$$S(\pi - x) = S(x), \qquad C(\pi - x) = -C(x)$$

$$S(2\pi + x) = S(x), \qquad C(2\pi + x) = C(x)$$

Thus, we see that the functions $C(x)$ and $S(x)$ exactly coincide with the functions cos x and sin x respectively, and so we shall henceforth use cos x and sin x in place of $C(x)$ and $S(x)$ respectively.

8.5.3 The Functions tan *x*, cot *x*

The function tan x and cot x are defined as usual by the ratios

$$\tan x = \frac{\sin x}{\cos x},\ \cot x = \frac{\cos x}{\sin x}$$

and as functions they, therefore, represent nothing new. The expansions in power series for these functions are also not so simple. A few of the coefficients of the expansions could be easily obtained by division, but that gives us no insight into any relationships.

Clearly tan x is defined, continuous and derivable for all values of x except those for which the denominator, cos x, vanishes, which is the case for $x = \frac{1}{2}(2n+1)\pi$, n being any integer, positive, negative or zero.

From § 4.2 (f), we have

$$\tan(\pi + x) = \tan x,$$

so that, tan x is a periodic function with period π.

Also we may easily show that when $x \neq \frac{1}{2}(2n+1)\pi$,

$$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{1}{\cos^2 x}$$

Theorem 8.2. *Show that*

$$\lim_{x \to \frac{1}{2}\pi - 0} \tan x = \infty, \quad \lim_{x \to \frac{1}{2}\pi + 0} \tan x = -\infty.$$

Let k be any positive number.

As $\lim_{x \to \pi/2} \sin x = 1$, $\exists\, \delta_1 > 0$, such that $\left(\text{taking } \varepsilon = \frac{1}{2}\right)$,

$$\frac{1}{2} < \sin x, \quad \forall\ x \in \left[\frac{1}{2}\pi - \delta_1, \frac{1}{2}\pi + \delta_1\right] \qquad ...(i)$$

Again, since, $\lim_{x \to \pi/2} \cos x = 0$, therefore, $\exists\, \delta_2 > 0$, such that

$$-\frac{1}{2k} < \cos x < \frac{1}{2k}, \quad \forall\ x \in \left[\frac{1}{2}\pi - \delta_2, \frac{1}{2}\pi + \delta_2\right]$$

As cos x is positive for $x \in [0, \pi/2[$, and negative for $x \in\ ]\pi/2, \pi]$, we have

$$0 < \cos x < \frac{1}{2k}, \quad \forall\ x \in \left[\frac{1}{2}\pi - \delta_2, \frac{1}{2}\pi\right[\qquad ...(ii)$$

$$-\frac{1}{2k} < \cos x < 0, \quad \forall\ x \in \left]\frac{1}{2}\pi, \frac{1}{2}\pi + \delta_2\right] \qquad ...(iii)$$

Let $\delta = \min(\delta_1, \delta_2)$ therefore from (i) and (ii),

$$\tan x = \frac{\sin x}{\cos x} > k, \quad \forall\ x \in \left[\frac{1}{2}\pi - \delta, \frac{1}{2}\pi\right[$$

and from (i) and (iii),

$$\tan x = \frac{\sin x}{\cos x} < -k, \quad \forall\ x \in \left]\frac{1}{2}\pi, \frac{1}{2}\pi + \delta\right]$$

8.5.4 Inverse Trigonometric Functions $\cos^{-1} y$, $\sin^{-1} y$, $\tan^{-1} y$

$\cos^{-1} y$. Since, as may be easily seen, $\cos x$ strictly decreases from +1 to –1 as x increases from 0 to π, the function cos is invertible and its inverse, denoted as $\cos^{-1}$, is a function with domain [–1, 1] and range $[0, \pi]$. We write

$$y = \cos x \Leftrightarrow x = \cos^{-1} y.$$

Definition. Given y (where $-1 \le y \le 1$), $\cos^{-1} y$ is that x which lies between 0 and $\pi\,(0 \le x \le \pi)$ and $\cos x = y$.

$\cos^{-1} y$ is derivable in the open interval]–1, 1[, with $-1/\sqrt{1 - y^2}$ as its derivative. In fact, we have

$\therefore$ $$\frac{dx}{dy} \cdot \frac{dy}{dx} = 1, \text{ and } x = \cos^{-1} y,\ y = \cos x$$

$$\frac{d}{dy}\left(\cos^{-1} y\right) = \frac{1}{\dfrac{d}{dx}\cos x} = -\frac{1}{\sin x} = \frac{-1}{\sqrt{1 - y^2}},\ y \ne \pm 1.$$

$\sin^{-1} y$. Since $\sin x$ is a strictly increasing function in $[-\pi/2, \pi/2]$, with range [–1, 1], therefore, the function sin is invertible and its inverse function is denoted by $\sin^{-1}$, with domain [–1, 1] and range $[-\pi/2, \pi/2]$.

Also

$$y = \sin x \Leftrightarrow x = \sin^{-1} y$$

Definition. Given y where $-1 \le y \le 1$, $\sin^{-1} y$ is that x which lies between $-\pi/2$ and $\pi/2$, $(-\pi/2 \le x \le \pi/2)$, and $\sin x = y$.

It may be shown as before that $\sin^{-1} y$ is derivable in the open interval]–1, 1[and

$$\frac{d}{dy}\sin^{-1} y = \frac{1}{\sqrt{1 - y^2}},\ y \ne \pm 1$$

$\tan^{-1} y$. Since $\tan x$ is strictly monotonic with domain $]-\pi/2, \pi/2[$ and range $]-\infty, \infty[$, the function is invertible, we have

$$y = \tan x \Leftrightarrow x = \tan^{-1} y$$

so that $\tan^{-1} y$ is a function with domain $]-\infty, \infty[$ and range $]-\pi/2, \pi/2[$.

Definition. For any number y, $\tan^{-1} y$ is that x which lies between $-\pi/2$ and $\pi/2\,(-\pi/2 < x < \pi/2)$ and $\tan x = y$.

It may be seen that

$$\frac{d}{dy}\tan^{-1} y = \frac{1}{1 + y^2},\ \forall\ y.$$

CHAPTER 9

The Riemann Integral

9.1 INTRODUCTION

At elementary stage, the subject of integration is generally introduced as the inverse of differentiation, so that a function F is called an integral of a given function f if $F'(x) = f(x)$, for all values of x belonging to the domain of the function f. Historically speaking the subject arose in connection with the evaluation of areas of plane regions and thus amounted to finding out the limit of a sum when the number of terms tended to infinity, each term tending to zero. Realisation that the subject could be looked upon as inverse of differentiation came afterwards. The reference to integration from summation point of view was always associated with the geometrical concepts.

To formulate an independent theory of integration, the German mathematician, Riemann, gave a purely arithmetic treatment to the subject and thus developed the subject entirely free from the intuitive dependence on geometrical concepts. Many refinements and generalisations of the subject followed, the most noteworthy being *Lebesgue theory* of integration.

The present chapter is, thus, based on a definition of the Riemann integral which depends very explicitly on the order structure of the real line. Accordingly, we begin by discussing the integration of real bounded functions on intervals. Integration over sets other than intervals is beyond the scope of the present discussion.

The function will always be assumed to be bounded unless otherwise stated.

9.2 DEFINITIONS AND EXISTENCE OF THE INTEGRAL

Let $[a, b]$ be a given closed interval.

Partition. By a partition of $[a, b]$ we mean a finite set P of points $x_0, x_1, x_2, ..., x_n$, where

$$a = x_0 \leq x_1 \leq ... \leq x_{n-1} \leq x_n = b$$

The partition P consists of $n + 1$ points. Clearly any number of partitions of $[a, b]$ can be considered.

$[x_0, x_1], [x_1, x_2], ..., [x_{i-1}, x_i], ..., [x_{n-1}, x_n]$ are the sub-intervals of $[a, b]$. We shall use the same symbol Δx_i to denote the ith sub-interval $[x_{i-1}, x_i]$ as also its length $x_i - x_{i-1}$. Thus,

$$\Delta x_i = x_i - x_{i-1} \quad (i = 1, 2, ..., n)$$

Let f be a *bounded real-valued* function on $[a, b]$. Evidently f is bounded on each sub-interval corresponding to each partition P. Let M_i, m_i be the bounds (supremum and infimum) of f in Δx_i.

From the two sums,

$$U(P,f) = \sum_{i=1}^{n} M_i \, \Delta x_i = M_1 \Delta x_1 + M_2 \Delta x_2 + ... + M_n \Delta x_n$$

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i = m_1 \Delta x_1 + m_2 \Delta x_2 + ... + m_n \Delta x_n$$

respectively called the *Upper* and the *Lower* (Darboux) *sums* of f corresponding to the partition P.

If M, m are the bounds of f in $[a, b]$, we have

$$m \le m_i \le M_i \le M$$

$$\Rightarrow m \Delta x_i \le m_i \Delta x_i \le M_i \Delta x_i \le M \Delta x_i$$

Putting $i = 1, 2, ..., n$ and adding all the inequalities, we get

$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a), \; b \ge a \qquad ...(1)$$

Now each partition gives rise to a pair of sums, the upper and the lower sums. By considering all partitions of $[a, b]$, we get a set U of upper sums and a set L of lower sums. The inequality (1) shows that both these sets are bounded and so each set has the supremum and the infimum. The *infimum* of the set of upper sums is called the *Upper Integral* and the *supremum* of the set of lower sums is called the *Lower Integral* of f over $[a, b]$. Thus,

$$\int_a^b f \, dx = \inf U \quad \text{or} \quad \inf \{U(P, f): P \text{ is a partition of } [a, b]\}$$

$$\int_a^b f \, dx = \sup L \quad \text{or} \quad \sup \{L(P, f): P \text{ is a partition of } [a, b]\}$$

These two integrals may or may not be equal.

Definition 1. (*Darboux's condition of integrability*). When the two integrals are equal, *i.e.*,

$$\bar{\int_a^b} f \, dx = \int_a^b f \, dx = \int_a^b f \, dx$$

we say that f is *Riemann Integrable* (or simply *integrable*) over $[a, b]$ and the common value of these integrals is called the *Riemann Integral* (or simply the *integral*) of f over $[a, b]$.

The fact that f is integrable, we express by writing

$$f \in \mathbf{R}$$

Evidently from (1),

$$m(b-a) \le \int_a^b f \, dx \le M(b-a), \;\; b \ge a \qquad ...(2)$$

Thus the upper and lower integrals are defined for *every bounded function* but they may not necessarily be equal for every bounded function. There exist functions for which these integrals are not equal; such functions are not integrable. Thus the question of their equality, and hence the question of the integrability of a function, is a more delicate one and will be our main concern in the next few pages.

Note. For the sake of convenience, whenever the scope for confusion is not there, we shall omit the limits of integration and write simply.

$$\bar{\int} f\,dx, \quad \int f\,dx, \quad \int f\,dx$$

Remarks 1. The statement that $\int_a^b f\,dx$ exists, implies that the function f is *bounded* and *integrable* over $[a, b]$.

2. We have introduced the concept of integrability of a function subject to two very important limitations, *viz.* (*i*) the function is bounded, (*ii*) the interval is finite.

3. From (1) and (2) it follows that when $b > a$,

$$m(b-a) \le L(P,f) \le \int_a^b f\,dx \le U(P,f) \le M(b-a)$$

4. Since the upper integral is the greatest lower bound of the set of upper sums, therefore, corresponding to any $\varepsilon_1 > 0$, $\exists$ an upper sum (or $\exists$ a partition P_1) such that

$$U(P_1 f) < \bar{\int_a^b} f\,dx + \varepsilon_1$$

Similarly

$$L(P_2 f) > \underline{\int_a^b} f\,dx - \varepsilon_2$$

5. $U(P,f) - L(P,f) = \sum_i M_i \Delta x_i - \sum_i m_i \Delta x_i = \sum_i (M_i - m_i)\,\Delta x_i$
$(M_i - m_i)$ being the oscillation of f in the sub-interval Δx_i, $U(P, f) - L(P, f)$ is called the *oscillatory sum* denoted by $\omega(P, f)$ and is non-negative.

Example 9.1. Show that a constant function k is integrable and

$$\int_a^b k\,dx = k(b-a)\,.$$

Solution. For any partition P of the interval $[a, b]$, we have

$$\begin{aligned} L(P,f) &= k\ \Delta x_1 + k\ \Delta x_2 + \ldots + k\ \Delta x_n \\ &= k\,(\Delta x_1 + \Delta x_2 + \ldots + \Delta x_n) \\ &= k(b-a) \end{aligned}$$

$$\begin{aligned} \Rightarrow \int_a^b k\,dr &= \sup L(P,f) \\ &= k(b-a) \end{aligned}$$

$$\begin{aligned} \bar{\int_a^b} k\,dx &= \inf\ U(P,f) \\ &= \inf\ (k\ \Delta x_1 + k\ \Delta x_2 + \ldots + k\ \Delta x_n) \\ &= k(b-a) \end{aligned}$$

Thus

$$\underline{\int}_a^b k\ dx = \bar{\int}_a^b k\,dx = k(b-a)$$

which implies that the function k is integrable and

$$\int_a^b k\,dx = k(b-a)$$

Example 9.2. Show that the function f defined by

$$f(x) = \begin{cases} 0, \text{ when } x \text{ is rational,} \\ 1, \text{ when } x \text{ is irrational} \end{cases}$$

is not integrable on any interval.

Solution. Let us consider a partition P of an interval $[a, b]$.

$$\begin{aligned} U(P, f) &= \sum_{i=1}^{n} M_i\ \Delta x_i \\ &= \Delta x_1 + 1\Delta x_2 + \ldots + \Delta x_n \\ &= b - a \end{aligned}$$

$\therefore$ $$\bar{\int}_a^b f\,dx = \inf\ U(P, f) = b - a$$

$$\begin{aligned} \underline{\int}_a^b f\,dx &= \sup\ L(P, f) \\ &= \sup\ \{0\Delta x_1 + 0\Delta x_2 + \ldots + 0\Delta x_n\} = 0 \end{aligned}$$

Thus

$$\bar{\int} f\,dx \neq \underline{\int} f\,dx$$

Hence, the function f is not integrable.

Example 9.3. Show that x^2 is integrable on any interval $[0, k]$.

Solution. Let us consider the partition P of $[0, k]$ obtained by dividing the interval into n equal parts. Thus, $[0, k/n, 2k/n, \ldots, nk/n]$ is the partition P, $[(i-1)(k/n)]^2$ and $[i(k/n)]^2$ are the lower and upper bounds of the function in Δx_i, and the length of each such interval is k/n.

$$U(P, x^2) = \frac{k^3}{n^3}(1^2 + 2^2 + 3^2 + \ldots + n^2)$$

$\therefore$ $$= \frac{k^3}{n^3} \cdot \frac{n}{6}(n+1)(2n+1)$$

$$= \frac{k^3}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right),$$

and

$$L(P, x^2) = \frac{k^3}{n^3}\{0 + 1^2 + 2^2 + ... + (n-1)^2\}$$

$$= \frac{k^3}{6}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)$$

$$\therefore \qquad \inf U(P, x^2) = \frac{k^3}{3} = \sup L(P, x^2)$$

Hence, the function x^2 is integrable and

$$\int_0^k x^2\, dx = k^3/3$$

EXERCISE

1. Show that $(3x + 1)$ is integrable on $[1, 2]$ and

$$\int_1^2 (3x + 1)\, dx = \frac{11}{2}$$

2. A function f is bounded on $[a, b]$. Show that
   (*i*) when k is a positive constant,

$$\bar{\int}_a^b kf\, dx = k\,\bar{\int}_a^b f\, dx; \quad \underline{\int}_a^b kf\, dx = k\,\underline{\int}_a^b f\, dx$$

   and (*ii*) when k is a negative constant,

$$\bar{\int}_a^b kf\, dx = k\,\underline{\int}_a^b f\, dx; \quad \underline{\int}_a^b kf\, dx = k\,\bar{\int}_a^b f\, dx$$

   Deduce that if f is bounded and integrable over $[a, b]$ then so is kf, where k is any constant, and then

$$\int_a^b kf\, dx = k\int_a^b f\, dx$$

[*Hint*. If M_i, m_i, are bounds of f in Δx_i, then kM_i, km_i, are bounds of kf in Δx_i, if k is positive. But if k is negative, the bounds of kf are km_i, kM_i].

9.2.1 Definition

The meaning of $\int_a^b f\, dx$ when $b \le a$. If f is bounded and integrable on $[b, a]$, for $a > b$, we define

$$\int_a^b f\, dx = -\int_b^a f\, dx, \text{ when } a > b$$

Also

$$\int_a^b f\,dx = 0, \text{ when } a = b$$

9.2.2 Inequalities for Integrals

In an earlier section we have proved that for a bounded integrable function f,

$$m(b-a) \le \int_a^b f\,dx \le M(b-a), \text{ when } b \ge a \qquad ...(3)$$

If $b < a$, so that $a > b$, then as proved above

$$m(a-b) \le \int_b^a f\,dx \le M(a-b), \text{ when } a > b$$

$$\Rightarrow -m(a-b) \ge -\int_b^a f\,dx \ge -M(a-b)$$

$$\Rightarrow m(b-a) \ge \int_a^b f\,dx \ge M(b-a), \text{ when } b < a \qquad ...(4)$$

We shall now make some interesting deductions from these two inequalities.

Deduction 1. If f is bounded and integrable on $[a, b]$, then there exists a number λ lying between the bounds of f, such that

$$\int_a^b f\,dx = \lambda(b-a)$$

Deduction 2. If f is continuous and integrable on $[a, b]$, then there exists a number c between a and b such that

$$\int_a^b f\,dx = (b-a)\,f(c)$$

Deduction 3. If f is bounded and integrable on $[a, b]$ and k is a number such that $|f(x)| \le k$, for all $x \in [a, b]$, then

$$\left|\int_a^b f\,dx\right| \le k\,|b-a|$$

Let M, m be the bounds of $f(x)$.

Now $\quad |f(x)| \le k, \ \forall\ x \in [a, b]$

$\therefore \quad -k \le f(x) \le k \Rightarrow -k \le m \le f(x) \le M \le k$

which for $b \ge a$, implies that

$$-k(b-a) \le m(b-a) \le \int_a^b f\,dx \le M(b-a) \le k(b-a)$$

$$\Rightarrow \left| \int_a^b f\,dx \right| \le k(b-a)$$

If $b < a$, so that $a > b$, we have

$$\left| \int_b^a f\,dx \right| \le k\,|a-b| \Rightarrow \left| \int_a^b f\,dx \right| \le k\,|b-a|$$

The result is trivial for $a = b$.

Deduction 4. If f is bounded and integrable on $[a, b]$, and $f(x) \ge 0$, for all $x \in [a, b]$, then

$$\int_a^b f\,dx \begin{cases} \ge 0, & \text{if } b \ge a \\ \le 0, & \text{if } b \le a \end{cases}$$

Since $f(x) \ge 0,$ for all $x \in [a, b],$ therefore, the lower bound $m \ge 0.$

The result follows from the inequalities (3) and (4) above.

Deduction 5. If f, g are bounded and integrable on $[a, b]$, such that $f \ge g,$ then

$$\int_a^b f\,dx \ge \int_a^b g\,dx, \text{ when } b \ge a$$

and

$$\int_a^b f\,dx \le \int_a^b g\,dx, \text{ when } b \le a$$

Now

$$f \ge g \Rightarrow f - g \ge 0, \quad \forall\ x \in [a, b]$$

$\therefore$ Using Deduction 4, we have

$$\int_a^b (f-g)\,dx \ge 0 \text{ if } b \ge a$$

or

$$\int_a^b f\,dx \ge \int_a^b g\,dx \text{ if } b \ge a$$

Similarly,

$$\int_a^b f\,dx \le \int_a^b g\,dx \text{ if } b \le a$$

Note. We have assumed the result $\int (f - g)\, dx = \int f\, dx - \int g\, dx$. It will however be proved in § 9.6, Theorem 9.6.

9.3 REFINEMENT OF PARTITIONS

Definition. For any partition P, the length of the largest sub-interval is called the *norm* or *mesh* of the partition and is denoted as $\mu(P)$, (or simply μ).

$$\therefore \qquad \mu(P) = \max \Delta x_i \ (1 \le i \le n)$$

A partition P^* is said to be a *refinement* of P if $P^* \supseteq P$, *i.e.*, every point of P is a point of P^*.

We also say that P^* *refines* P or that P^* is *finer than* P.

If P_1 and P_2 are two partitions, then we say that P^* is their *common refinement* if $P^* = P_1 \cup P_2$.

Theorem 9.1. *If P^* is a refinement of a partition P, then for a bounded function f,*

(*i*) $L(P^*, f) \ge L(P, f)$, *and*

(*ii*) $U(P^*, f) \le U(P, f)$.

To prove (*i*), suppose first that P^* contains just one point more than P.

Let this extra point be ξ, and suppose that this point is in Δx_i, that is, $x_{i-1} < \xi < x_i$.

As the function is bounded over the entire interval $[a, b]$, it is bounded in every sub-interval Δx_i $(i = 1, 2, ..., n)$. Let w_1, w_2, m_i be the infimum (g.l.b.) of f in the intervals $[x_{i-1}, \xi]$, $[\xi, x_i]$, $[x_{i-1}, x_i]$, respectively.

Clearly $m_i \le w_1, m_i \le w_2$.

$$\begin{aligned} \therefore \qquad & L(P^*, f) - L(P, f) \\ & = w_1(\xi - x_{i-1}) + w_2(x_i - \xi) + m_i(x_i - x_{i-1}) \\ & = (w_1 - m_i)(\xi - x_{i-1}) + (w_2 - m_i)(x_i - \xi) \\ & \ge 0 \quad (\because \text{ each bracket is positive}) \end{aligned}$$

If P^* contains p points more than P, we repeat the above reasoning p times and arrive at

$$L(P^*, f) \ge L(P, f)$$

Similarly, we can prove that

$$U(P^*, f) \le U(P, f)$$

Corollary. If a refinement P^* of P contains p points more than P, and $|f(x)| \le k$, *for all* $x \in [a, b]$, then

$$L(P, f) \le L(P^*, f) \le L(P, f) + 2pk\mu$$

$$U(P, f) \ge U(P^*, f) \ge U(P, f) - 2pk\mu$$

Proceeding as in the above theorem, when P^* contains one point more than P, we get

$$L(P^*, f) - L(P, f) = (w_1 - m_i)(\xi - x_{i-1}) + (w_2 - m_i)(x_i - \xi)$$

Since $|f(x)| \le k$, for all $x \in [a, b]$, therefore,

$$\begin{aligned} & -k \le m_i \le w_1 \le k \\ \Rightarrow \quad & 0 \le w_1 - m_i \le 2k \end{aligned}$$

Similarly

$$0 \le w_2 - m_i \le 2k$$

$$\therefore \quad L(P^*, f) - L(P, f) \le 2k(\xi - x_{i-1}) + 2k(x_i - \xi)$$

$$= 2k\ \Delta x_i$$

$$\le 2k\mu, \text{ where } \mu \text{ is norm of } P$$

Now supposing that each additional point is introduced one by one, by repeating the above reasoning p times we get

$$L(P^*, f) \le L(P, f) + 2pk\mu$$

Also

$$L(P, f) \le L(P^*, f)$$

$$\therefore \quad L(P, f) \le L(P^*, f) \le L(P, f) + 2pk\mu$$

Similarly the other result may be proved.

Exercise. If P^* is a refinement of P then

$$U(P^*, f) - L(P^*, f) \le U(P, f) - L(P, f)$$

Theorem 9.2. *For any two partitions* P_1, P_2,

$$L(P_1, f) \le U(P_2, f)$$

i.e., no upper sum can ever be less than any lower sum.

Let P^* be a common refinement of P_1, P_2, so that

$$P^* = P_1 \cup P_2$$

Using Theorem 1, we get

$$L(P_1, f) \le L(P^*, f) \le U(P^*, f) \le U(P_2, f) \qquad ...(1)$$

$$\therefore \quad L(P_1, f) \le U(P_2, f)$$

Corollary. For any bounded function f,

$$\underline{\int} f\, dx \le \overline{\int} f\, dx$$

By keeping P_2 fixed and taking the l.u.b. over all partitions P_1, equation (1) of Theorem 9.2 gives

$$\underline{\int} f\, dx \le U(P_2, f) \qquad ...(2)$$

Taking the g.l.b. over all P_2 in (2), we get

$$\underline{\int} f\, dx \le \overline{\int} f\, dx\,.$$

9.4 DARBOUX'S THEOREM

If f is a bounded function on $[a, b]$, *then to every* $\varepsilon > 0$, *there corresponds* $\delta > 0$, *such that*

(*i*) $U(P, f) < \bar{\int}_a^b f\,dx + \varepsilon,$

(*ii*) $L(P, f) > \underline{\int}_a^b f\,dx - \varepsilon,$

for every partition P of $[a, b]$, *with norm* $\mu(P) < \delta$.

Let us first attend to (*i*).

As f is bounded, therefore, $\exists\ k > 0,$ such that

$$|f(x)| \le k, \quad \forall\ x \in [a, b]$$

Again, since the upper integral is the infimum (g.l.b.) of the set of upper sums, therefore to every $\varepsilon > 0$ there exists a partition $P_1 = \{x_0, x_1, x_2, ..., x_p\}$ of $[a, b]$, such that

$$U(P_1, f) < \bar{\int}_a^b f\,dx + \frac{1}{2}\varepsilon \qquad ...(1)$$

The partition P_1 has $p - 1$ points besides $x_0 (= a)$ and $x_p (= b)$. Let δ be a positive number, such that

$$2k(p - 1)\delta = \frac{1}{2}\varepsilon. \qquad ...(2)$$

Let P be any partition with norm $\mu(P) < \delta$.

Let, further, P^* be a refinement of P and P_1, so that $P^* = P \cup P_1$.

As P^* is a refinement of P having at the most $p - 1$ more points than P, therefore, (using Corollary, Theorem 9.1) we get

$$\begin{aligned} U(P, f) - 2k(p - 1)\delta &\le U(P^*, f) \\ &\le U(P_1, f) \\ &< \bar{\int}_a^b f\,dx + \frac{1}{2}\varepsilon \quad \text{(using (1))} \end{aligned}$$

$$\Rightarrow \qquad U(P, f) < \bar{\int}_a^b f\,dx + \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \int_a^b f\,dx + \varepsilon \quad \text{(using (2))}$$

for any partition P with norm $\mu(P) < \delta$.

Similarly we can prove the other part.

Note. The definition of infimum also leads to the fact that

$$U(P, f) < \bar{\int}_a^b f\,dx + \varepsilon$$

but this implies that for every $\varepsilon > 0$ there exists at least one partition P with this property. The importance of Darboux's Theorem lies in the fact that it asserts the existence of an infinite number of partitions which have this property, with the only restriction that their norm $\mu(P) < \delta,$ depending on the choice of ε.

9.5 CONDITIONS OF INTEGRABILITY

We have stated earlier that a bounded function is said to be integrable when the upper and the lower integrals are equal. We now formalise and give the necessary and sufficient conditions for the integrability of a function in two forms.

Theorem 9.3. (*First form*). *A necessary and sufficient condition for the integrability of a bounded function f is that to every* $\varepsilon > 0$, *there corresponds* $\delta > 0$ *such that for every partition P of* $[a, b]$ *with norm* $\mu(P) < \delta$,

$$U(P, f) - L(P, f) < \varepsilon.$$

The condition is necessary. The bounded function f is integrable,

$$\therefore \qquad \underline{\int_a^b} f\,dx = \bar{\int_a^b} f\,dx = \int_a^b f\,dx$$

Let ε be any positive number. By Darboux's Theorem there exists $\delta > 0$ such that for every partition P with norm $\mu(P) < \delta$,

$$U(P, f) < \bar{\int_a^b} f\,dx + \frac{1}{2}\varepsilon = \int_a^b f\,dx + \frac{1}{2}\varepsilon \qquad \ldots(1)$$

$$L(P, f) > \int_a^b f\,dx - \frac{1}{2}\varepsilon = \int_a^b f\,dx - \frac{1}{2}\varepsilon \qquad \ldots(2)$$

or

$$-L(P, f) < -\int_a^b f\,dx + \frac{1}{2}\varepsilon \qquad \ldots(3)$$

From (1) and (3), we get on adding

$$U(P, f) - L(P, f) < \varepsilon \qquad \ldots(4)$$

for every partition P with norm $\mu(P) < \delta$.

The condition is sufficient. Let ε be any positive number. For any partition P with norm $\mu(P) < \delta$ (depending on ε), we are given that

$$U(P, f) - L(P, f) < \varepsilon$$

Also for any partition P, we know that

$$L(P,f) \le \underline{\int_a^b} f\,dx \le \bar{\int_a^b} f\,dx \le U(P,f)$$

$$\Rightarrow \bar{\int_a^b} f\,dx - \underline{\int_a^b} f\,dx \le U(P,f) - L(P,f) < \varepsilon$$

Since ε is an arbitrary positive number, therefore, we see that a non-negative number is less than every positive number. Hence, it must be equal to zero,

$$\Rightarrow \bar{\int_a^b} f\,dx - \underline{\int_a^b} f\,dx = 0$$

$$\Rightarrow \bar{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx,$$

so that f is integrable.

Note. The theorem can also be stated as follows:
A necessary and sufficient condition for the integrability of a bounded function f is that
$$\lim \{U(P,f) - L(P,f)\} = 0$$
when the norm $\mu(P)$ of the partition P tends to 0.

Theorem 9.4. (*Second form*). *A bounded function f is integrable on $[a, b]$ iff for every $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that*
$$U(P,f) - L(P,f) < \varepsilon$$

The condition is necessary. Suppose the function f is integrable, so that
$$\int_{\underline{a}}^{b} f\,dx = \bar{\int}_{a}^{b} f\,dx = \int_{a}^{b} f\,dx$$

Let ε be a positive number.

Since the upper and the lower integrals are the infimum and the supremum respectively of the upper and the lower sums, therefore $\exists$ partitions P_1 and P_2 such that
$$U(P_1,f) < \bar{\int}_{a}^{b} f\,dx + \frac{1}{2}\varepsilon = \int_{a}^{b} f\,dx + \frac{1}{2}\varepsilon$$

$$L(P_2,f) > \int_{\underline{a}}^{b} f\,dx - \frac{1}{2}\varepsilon = \int_{a}^{b} f\,dx - \frac{1}{2}\varepsilon$$

Let P be the common refinement of P_1 and P_2, *i.e.*, $P = P_1 \cup P_2$,

$$\therefore \quad U(P,f) \le U(P_1,f) < \int_{a}^{b} f\,dx + \frac{1}{2}\varepsilon < L(P_2,f) + \varepsilon \le L(P,f) + \varepsilon$$

Thus $\exists$ a partition P such that
$$U(P,f) - L(P,f) < \varepsilon$$

The condition is sufficient. Let ε be any positive number.

Let P be a partition for which
$$U(P,f) - L(P,f) < \varepsilon$$

For any partition P, we know that
$$L(P,f) \le \int_{\underline{a}}^{b} f\,dx \le \bar{\int}_{a}^{b} f\,dx \le U(P,f)$$

$$\therefore \quad \bar{\int}_{a}^{b} f\,dx - \int_{\underline{a}}^{b} f\,dx \le U(P,f) - L(P,f) < \varepsilon$$

The non-negative number, being less than every positive number ε, must be zero.

$$\therefore \quad \bar{\int}_{a}^{b} f\,dx = \int_{\underline{a}}^{b} f\,dx$$

so that f is integrable.

Note. Comparison of the two forms indicates that from the necessary point of view, the first form is stronger than the second but from the sufficiency view point the second form is stronger than the first.

9.5.1 Deduction 6

A function f is integrable over $[a, b]$ *iff there is a number I lying between* $L(P, f)$ *and* $U(P, f)$ *such that for any* $\varepsilon > 0, \exists$ *a partition P of* $[a, b]$ *such that*

$$|U(P, f) - I| < \varepsilon, \text{ and } |I - L(P, f)| < \varepsilon$$

Necessary: As $f \in \mathbf{R}[a, b]$, therefore for $\varepsilon > 0, \exists$ a partition P of $[a, b]$ such that

$$|U(P, f) - L(P, f)| < \varepsilon$$

If I is a number between $L(P, f)$ and $U(P, f)$, then

$$|U(P, f) - I| < |U(P, f) - L(P, f)| < \varepsilon$$

and

$$|I - L(P, f)| < |U(P, f) - L(P, f)| < \varepsilon$$

Hence, the result.

Sufficient: For $\varepsilon > 0, \exists$ a partition P such that

$$|U(P, f) - I| < \frac{1}{2}\varepsilon \text{ and } |I - L(P, f)| < \frac{1}{2}\varepsilon$$

$$\therefore \quad |U(P, f) - L(P, f)| \le |U(P, f) - I| + |I - L(P, f)|$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

$$\Rightarrow \quad f \in \mathbf{R}[a, b]$$

Deduction 7. A function f is integrable over $[a, b]$ iff there is a number I such that for any $\varepsilon > 0, \exists\, \delta$ such that for all partitions P with mesh $\mu(P) < \delta$,

$$|U(P, f) - I| < \varepsilon, \text{ and } |I - L(P, f)| < \varepsilon$$

The proof is similar to that of deduction 6 and is left to the reader.

Note. We know that $U(P^*, f) < U(P, f)$ when $P^* \supset P$. Therefore, the upper sum becomes smaller and smaller as the partition gets finer and finer. Thus, the upper integral $\bar{\int}_a^b f\,dx$, which is the g.l.b. of the set U of upper sums, can be looked upon as $\lim_{\mu(P)\to 0} U(P, f)$ or that for $\varepsilon > 0\ \exists\, \delta > 0$ such that for all partitions P with norm $\mu(P) < \delta, |U(P, f) - \bar{\int} f\,dx| < \varepsilon$. Thus Deduction 7 merely states that the upper and the lower sums converge to the same quantity I and moreover $I = \int_a^b f\,dx$.

9.6 INTEGRABILITY OF THE SUM AND DIFFERENCE OF INTEGRABLE FUNCTIONS

Theorem 9.5. *If f_1 and f_2 are two bounded and integrable functions on $[a, b]$ then $f = f_1 + f_2$ is also integrable on $[a, b]$, and*

$$\int_a^b f_1\,dx + \int_a^b f_2\,dx = \int_a^b f\,dx.$$

Clearly f is bounded on $[a, b]$.

Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be any partition of $[a, b]$ and $M_i', m_i'; M_i'', m_i''; M_i, m_i,$ be the bounds of f_1, f_2 and f respectively in Δx_i.

$[M_i' + M_i''$ and $m_i' + m_i''$ are rough upper and rough lower bounds while M_i and m_i the supremum and the infimum of f in $\Delta x_i.]$

$$m_i' + m_i'' \le m_i \le M_i \le M_i' + M_i'' \qquad ...(1)$$

Multiplying by Δx_i and adding all these inequalities for $i = 1,2,3,\ldots n,$ we get

$$L(P, f_1) + L(P, f_2) \le L(P, f) \le U(P, f) \le U(P, f_1) + U(P, f_2) \qquad ...(2)$$

Let ε be any positive number.

Since f_1, f_2 are integrable therefore we can choose $\delta > 0$ such that, for any partition P with norm $\mu(P) < \delta,$ we have

$$\left.\begin{aligned} U(P, f_1) - L(P, f_1) &< \frac{1}{2}\varepsilon \\ U(P, f_2) - L(P, f_2) &< \frac{1}{2}\varepsilon \end{aligned}\right\} \qquad ...(3)$$

Thus for any partition P with norm $\mu(P) < \delta$ we have, from (2) and (3),

$$U(P, f) - L(P, f) \le U(P, f_1) + U(P, f_2) - L(P, f_1) - L(P, f_2)$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

Thus the function f is integrable.

Now we proceed to prove the second part.

Since f_1, f_2 are integrable and ε is any positive number, therefore, by Darboux's Theorem, $\exists\ \delta > 0$ such that for all partitions P whose norm $\mu(P) < \delta,$ we have

and

$$\left.\begin{aligned} U(P, f_1) &< \int_a^b f_1\,dx + \frac{1}{2}\varepsilon, \\ U(P, f_2) &< \int_a^b f_2\,dx + \frac{1}{2}\varepsilon \end{aligned}\right\} \qquad ...(4)$$

Also

$$\int_a^b f\,dx \le U(P,f) \le U(P,f_1) + U(P,f_2) \qquad \text{...(using (2))}$$

$$< \int_a^b f_1\,dx + \int_a^b f_2\,dx + \varepsilon \qquad \text{...(using (4))}$$

Since ε is arbitrary, we conclude that

$$\int_a^b f\,dx \le \int_a^b f_1\,dx + \int_a^b f_2\,dx \qquad \text{...(5)}$$

Proceeding with $(-f_1), (-f_2)$ in place of f_1, f_2 respectively, we get

$$\int_a^b (-f)\,dx \le \int_a^b (-f_1)\,dx + \int_a^b (-f_2)\,dx$$

or

$$\int_a^b f\,dx \ge \int_a^b f_1\,dx + \int_a^b f_2\,dx \qquad \text{...(6)}$$

(5) and (6) give

$$\int_a^b f\,dx = \int_a^b f_1\,dx + \int_a^b f_2\,dx\,.$$

Note. When f_1 is integrable, for $\varepsilon > 0, \exists\, \delta_1 > 0$ such that for $\mu(P) < \delta_1, U(P,f) - L(P,f_1) < \frac{1}{2}\varepsilon$. Similarly $\exists\, \delta_2 > 0$ for the functions f_2. However, $\delta = \min(\delta_1, \delta_2)$ works for both the functions. It is this δ which was used for (3) above, δ for (4) was also selected by a similar reasoning.

Exercise. Prove the above theorem by using Theorem 9.4 for condition of integrability.

Theorem 9.6. If f_1, f_2 *are two bounded and integrable functions on* $[a, b]$, *then* $f = f_1 - f_2$ *is also integrable on* $[a, b]$ *and*

$$\int_a^b f\,dx = \int_a^b f_1\,dx - \int_a^b f_2\,dx\,.$$

Let $f = f_1 + (-f_2)$, so that f is bounded on $[a, b]$.

Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be any partition of $[a, b]$ and $M_i', m_i'; M_i'', m_i''; M_i, m_i$ be the bounds of f_1, f_2 and f respectively in Δx_i. Clearly the bounds of $(-f_2)$ are $-m_i''$ and $-M_i''$.

$$\therefore \qquad m_i' - M_i'' \le m_i \le M_i \le M_i' - m_i'' \qquad \text{...(7)}$$

Multiplying by Δx_i and adding all these inequalities for $i = 1, 2, \ldots, n$, we get

$$L(P,f_1) - U(P,f_2) \le L(P,f) \le U(P,f) \le U(P,f_1) - L(P,f_2) \qquad \text{...(8)}$$

Let $\varepsilon > 0$ be a given number.

Since f_1, f_2 are integrable, therefore $\exists\ \delta > 0$ such that for any partition P whose norm $\mu(P) < \delta$ we have

$$\left.\begin{aligned} U(P, f_1) - L(P, f_1) &< \frac{1}{2}\varepsilon \\ U(P, f_2) - L(P, f_2) &< \frac{1}{2}\varepsilon \end{aligned}\right\} \quad ...(9)$$

Thus for any partition P with norm $\mu(P) < \delta$, we have from (8) and (9),

$$U(P, f) - L(P, f) \le U(P, f_1) - L(P, f_2) - L(P, f_1) + U(P, f_2)$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon\ .$$

Thus the function f is integrable.

Let us now prove the second part.

Since f_1, f_2 are integrable and ε is any positive number, therefore, by Darboux's Theorem $\exists\ \delta > 0$ such that for all partitions P with norm $\mu(P) < \delta$, we have

$$U(P, f_1) < \int_a^b f_1\, dx + \frac{1}{2}\varepsilon, \quad \text{and} \quad L(P, f_2) > \int_a^b f_2\, dx - \frac{1}{2}\varepsilon \quad ...(10)$$

$$\therefore \quad \int_a^b f\, dx \le U(P, f)$$

$$\le U(P, f_1) - L(P, f_2) \quad ...\text{using (8)}$$

$$< \int_a^b f_1\, dx - \int_a^b f_2\, dx + \varepsilon$$

Since ε is arbitrary, we conclude that

$$\int_a^b f\, dx \le \int_a^b f_1\, dx - \int_a^b f_2\, dx$$

Proceeding with $(-f_1)$ and $(-f_2)$ in place of f_1 and f_2 respectively, we get

$$\int_a^b f\, dx \ge \int_a^b f_1\, dx - \int_a^b f_2\, dx$$

$$\therefore \quad \int_a^b f\, dx = \int_a^b f_1\, dx - \int_a^b f_2\, dx$$

Exercise. Prove the above theorem by using Theorem 9.4.

Theorem 9.7. (*i*) *If a bounded function f is integrable on* [*a*, *b*], *then it is also integrable on* [*a*, *c*] *and* [*c*, *b*], *where c is a point of* [*a*, *b*].

(*ii*) *Conversely, if f is bounded and integrable on* [*a*, *c*], [*c*, *b*] *then it is also integrable on* [*a*, *b*].

(*iii*) *Also in either case*,

$$\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx,\quad a \le c \le b$$

(*i*) Since $f \in \mathbf{R}\,[a, b]$, therefore for $\varepsilon > 0, \exists$ a partition P such that

$$U(P, f) - L(P, f) < \varepsilon$$

Let P^* be a refinement of P such that

$$P^* = P \cup \{c\}$$

$$\therefore \quad L(P, f) \le L(P^*, f) \le U(P^*, f) \le U(P, f) \qquad ...(1)$$

$$\Rightarrow \quad U(P^*, f) - L(P^*, f) \le U(P, f) - L(P, f) < \varepsilon \qquad ...(2)$$

Let P_1, P_2 denote the sets of points of P^* between $[a, c]$, $[c, b]$ respectively. Clearly P_1, P_2 are partitions of $[a, c]$, $[c, b]$ respectively and $P^* = P_1 \cup P_2$.

Also

$$U(P^*, f) = U(P_1, f) + U(P_2, f) \qquad ...(3)$$

and

$$L(P^*, f) = L(P_1, f) + L(P_2, f) \qquad ...(4)$$

$$\therefore \{U(P_1, f) - L(P_1, f)\} + \{U(P_2, f) - L(P_2, f)\} = U(P^*, f) - L(P^*, f) < \varepsilon$$

Since each bracket on the left is non-negative, it follows that partitions P_1, P_2 exist such that

$$U(P_1, f) - L(P_1, f) < \varepsilon,$$
$$U(P_2, f) - L(P_2, f) < \varepsilon.$$

$\Rightarrow f$ is integrable on $[a, c]$ and $[c, b]$

(*ii*) Let $f \in \mathbf{R}$ over $[a, c]$ and $[c, b]$.

$\therefore$ for $\varepsilon > 0$ we can find partitions P_1, P_2 of $[a, c]$, $[c, b]$ respectively such that

$$U(P_1, f) - L(P_1, f) < \frac{1}{2}\varepsilon \qquad ...(5)$$

$$U(P_2, f) - L(P_2, f) < \frac{1}{2}\varepsilon \qquad ...(6)$$

Let $P^* = P_1 \cup P_2$.

Clearly P^* is a partition of $[a, b]$.

$$\therefore \quad U(P^*, f) - L(P^*, f) = U(P_1, f) + U(P_2, f) - L(P_1, f) - L(P_2, f)$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

Thus for $\varepsilon > 0, \exists$ a partition P^* of $[a, b]$ such that

$$U(P^*, f) - L(P^*, f) < \varepsilon$$

$$\Rightarrow f \in \mathbf{R} \text{ over } [a, b]$$

(*iii*) We know that for any functions S and T, if $W = S + T$, then $\inf W > \inf S + \inf T$

Now for any partitions P_1, P_2 of $[a, c]$, $[c, b]$ respectively, if $P^* = P_1 \cup P_2$, then

$$U(P^*, f) = U(P_1, f) + U(P_2, f)$$

$\therefore$ on taking the infimum for all partitions, we get

$$\int_a^b f\,dx \geq \bar{\int_a^c} f\,dx + \bar{\int_c^b} f\,dx$$

But since f is integrable on $[a, c]$, $[c, b]$, $[a, b]$,

$$\therefore \quad \int_a^b f\,dx \geq \int_a^c f\,dx + \int_c^b f\,dx \qquad \ldots(7)$$

Proceeding with $(-f)$ in place of f, we get

$$\int_a^b f\,dx \leq \int_a^c f\,dx + \int_c^b f\,dx \qquad \ldots(8)$$

Equations (7) and (8) imply that

$$\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx\,.$$

Exercise. Do the above theorem by using Theorem 9.3 for conditions of integrability.

9.6.1* Integrability of the Product, Quotient and the Modulus of Integrable Functions

Before taking up the main theorems, let us prove a simple but useful lemma.

Lemma. *The oscillation of a bounded function f on an interval* $[a, b]$ *is the supremum of the set* $\{|f(x_1) - f(x_2)| : x_1, x_2 \in [a, b]\}$ *of numbers*.

Let M, m be the bounds of f, on $[a, b]$.

Now $\quad m \leq f(x_1), f(x_2) \leq M, \ \forall\ x_1, x_2 \in [a, b]$

$\Rightarrow \quad |f(x_1) - f(x_2)| \leq M - m, \ \forall\ x_1, x_2 \in [a, b] \qquad \ldots(1)$

$\Rightarrow \quad M - m$ is an upper bound of the set in Lemma.

Let $\varepsilon > 0$ be any given number.

Since M is supremum of f, therefore $\exists\ x' \in [a, b]$ such that

$$f(x') > M - \frac{1}{2}\varepsilon \qquad \ldots(2)$$

Similarly $\exists\ x'' \in [a, b]$ such that

$$f(x'') < m + \frac{1}{2}\varepsilon \qquad \ldots(3)$$

(2) and (3) imply that $\exists\ x', x'' \in [a, b]$ such that

$$f(x') - f(x'') > M - m - \varepsilon$$

$$\Rightarrow |f(x') - f(x'')| > M - m - \varepsilon \qquad \ldots(4)$$

(1) and (4) imply that $M - m$ is an upper bound and no number less than $M - m$ can be an upper bound of the set in Lemma.

$$\therefore \quad M - m = \sup\{|f(x_1) - f(x_2)| : x_1, x_2 \in [a, b]\}$$

Theorem 9.8*. *If* f_1 *and* f_2 *are two bounded and integrable functions on* $[a, b]$, *then their product* $f_1 f_2$ *is also bounded and integrable on* $[a, b]$.

Since f_1, f_2 are bounded therefore, $\exists\ k > 0$, such that, for all $x \in [a, b]$,

$$|f_1(x)| \le k, |f_2(x)| \le k$$

$$\Rightarrow \quad |(f_1 f_2)(x)| = |f_1(x)\, f_2(x)| \le k^2, \quad \forall\ x \in [a, b]$$

$\Rightarrow$ $f_1 \cdot f_2$ is bounded

Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be any partition of $[a, b]$.

Let M_i', m_i'; M_i'', m_i''; M_i, m_i be the bounds of f_1, f_2 and $f_1 f_2$ respectively in Δx_i.

We have for all $x_1, x_2 \in \Delta x_i$,

$$(f_1 f_2)(x_2) - (f_1 f_2)(x_1) = f_1(x_2) f_2(x_2) - f_1(x_1) f_2(x_1)$$
$$= f_2(x_2)[f_1(x_2) - f_1(x_1)] + f_1(x_1)[f_2(x_2) - f_2(x_1)]$$

$$\Rightarrow \quad |(f_1 f_2)(x_2) - (f_1 f_2)(x_1)| \le |f_2(x_2)| \cdot |f_1(x_2) - f_1(x_1)| + |f_1(x_1)| \cdot |f_2(x_2) - f_2(x_1)|$$
$$\le k(M_i' - m_i') + k(M_i'' - m_i'')$$

$$\Rightarrow \quad M_i - m_i \le k(M_i' - m_i') + k(M_i'' - m_i'') \qquad \ldots(1)$$

Now let $\varepsilon > 0$ be a given number.

Since f_1, f_2 are integrable, therefore $\exists\ \varepsilon > 0$ such that for any partition P with norm $\mu(P) < \delta$,

$$U(P, f_1) - L(P, f_1) < \varepsilon/2k, \quad U(P, f_2) - L(P, f_2) < \varepsilon/2k$$

$\therefore$ from (1), by multiplying by Δx_i and adding all such inequalities, we have for any partition P with norm $\mu(P) < \delta$,

$$U(P, f_1 f_2) - L(P, f_1 f_2) \le k[U(P, f_1) - L(P, f_1)] + k[U(P, f_2) - L(P, f_2)]$$
$$< k(\varepsilon/2k) + k(\varepsilon/2k) = \varepsilon$$

implying that $f_1\ f_2$ is integrable on $[a, b]$.

Theorem 9.9. *If* f_1, f_2 *are two bounded and integrable functions on* $[a, b]$ *and there exists a number* $\lambda > 0$, *such that* $|f_2(x)| \ge \lambda$, *for all x in* $[a, b]$, *then* f_1/f_2 *is bounded and integrable on* $[a, b]$.

Since f_1, f_2 are bounded, therefore $\exists$ positive number k such that

$$|f_1(x)| \le k, \lambda \le |f_2(x)| \le k, \quad \forall\ x \in [a, b]$$
$$\Rightarrow |(f_1/f_2)(x)| = |f_1(x)/f_2(x)| \le k/\lambda, \quad \forall\ x \in [a, b]$$
$$\Rightarrow f_1/f_2 \text{ is bounded}$$

Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be a partition of $[a, b]$ and let M_i', m_i'; M_i'', m_i''; m_i, M_i be the bounds of $f_1, f_2, f_1/f_2$, respectively in Δx_i. We have for all $x_1, x_2 \in \Delta x_i$,

$$|(f_1/f_2)(x_2) - (f_1/f_2)(x_1)| = \left| \frac{f_1(x_2)}{f_2(x_2)} - \frac{f_1(x_1)}{f_2(x_1)} \right|$$

$$= \left| \frac{f_2(x_1)[f_1(x_2) - f_1(x_1)] - f_1(x_1)[f_2(x_2) - f_2(x_1)]}{f_2(x_2)\ f_2(x_1)} \right|$$

$$\leq (k/\lambda^2)|f_1(x_2) - f_1(x_1)| + (k/\lambda^2)|f_2(x_2) - f_2(x_1)|$$

$$\leq (k/\lambda^2)(M_i' - m_i') + (k/\lambda^2)(M_i'' - m_i'')$$

$$\Rightarrow \quad M_i - m_i \leq (k/\lambda^2)(M_i' - m_i') + (k/\lambda^2)(M_i'' - m_i'') \qquad \ldots(1)$$

Now, let $\varepsilon > 0$ be a given number.

Since f_1, f_2 are integrable, therefore $\exists\ \delta > 0$ such that for any partition P with norm $\mu(P) < \delta$,

$$U(P, f_1) - L(P, f_1) < \varepsilon\lambda^2/2k$$

$$U(P, f_2) - L(P, f_2) < \varepsilon\lambda^2/2k$$

Hence from (1), for any partition P with $\mu(P) < \delta$, we have

$$U(P, f_1/f_2) - L(P, f_1/f_2) \leq (k/\lambda^2)\,[U(P, f_1) - L(P, f_1)]$$

$$+ (k/\lambda^2)\,[U(P, f_2) - L(P, f_2)]$$

$$< (k/\lambda^2)(\varepsilon\lambda^2/2k) + (k/\lambda^2)(\varepsilon\lambda^2/2k) = \varepsilon$$

implying that f_1/f_2 is integrable on $[a, b]$.

Theorem 9.10. *If f is bounded and integrable on $[a, b]$, then $|f|$ is also bounded and integrable on $[a, b]$. Moreover*

$$\left| \int_a^b f\,dx \right| \leq \int_a^b |f|\,dx.$$

Since f is bounded, therefore $\exists\ k > 0$, such that

$$|f(x)| \leq k, \quad \forall\ x \in [a, b]$$

$\Rightarrow$ the function $|f|$ is bounded.

Again, since f is integrable, $\exists$ a partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$ of $[a, b]$ such that

$$U(P, f) - L(P, f) < \varepsilon$$

Let $M_i, m_i;\ M_i', m_i'$ be the bounds of f and $|f|$ in Δx_i.

We have for all $x_1, x_2 \in \Delta x_i$

$$||f|(x_2) - |f|(x_1)| = ||f(x_2)| - |f(x_1)|| \leq |f(x_2) - f(x_1)| \leq M_i - m_i$$

$$\Rightarrow \quad M_i' - m_i' \leq M_i - m_i$$

This implies that for any partition P,

$$U(P, |f|) - L(P, |f|) \leq U(P, f) - L(P, f) < \varepsilon$$

Hence $|f|$ is integrable on $[a, b]$.

Since $f(x), -f(x) \leq |f(x)| = |f|(x)$, for all x in $[a, b]$, therefore by Deduction 5, we have

$$\int_a^b f\,dx \leq \int_a^b |f|\,dx$$

and

$$-\int_a^b f\,dx = \int_a^b (-f)\,dx \le \int_a^b |f|\,dx$$

$$\Rightarrow \quad \left|\int_a^b f\,dx\right| \le \int_a^b |f|\,dx$$

Note. The converse of the above theorem is not true. Consider, for example, the function,

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is rational} \\ -1, & \text{when } x \text{ is irrational} \end{cases}$$

Here

$$\overline{\int_a^b} f\,dx = b - a, \quad \underline{\int_a^b} f\,dx = -(b-a)$$

$\Rightarrow f$ is not integrable

But $|f(x)| = 1$, for all x, so that $\int_a^b |f|\,dx$ exists and is equal to $(b-a)$.

Thus $|f|$ is integrable while f is not.

Theorem 9.11. *If f is integrable on $[a, b]$, then f^2 is also integrable on $[a, b]$.*

Since f is bounded on $[a, b]$, therefore $|f|$ is also bounded on $[a, b]$.

Thus $\exists\ M > 0$, such that $|f(x)| \le M$, for all x in $[a, b]$.

Again, since f is integrable, therefore $|f|$ is also integrable on $[a, b]$, and therefore for $\varepsilon > 0$, $\exists$ a partition P of $[a, b]$ such that

$$U(P, |f|) - L(P, |f|) < \frac{\varepsilon}{2M}$$

Again, since $|f^2(x)| = |f(x)|^2 \le M^2$, therefore f^2 is bounded.

If M_i, m_i be the bounds of $|f|$ and M_i', m_i' those of f^2 in Δx_i then $M_i' = M_i^2, m_i' = m_i^2$. Also

$$\begin{aligned} U(P, f^2) - L(P, f^2) &= \sum_{i=1}^n (M_i' - m_i')\,\Delta x_i \\ &= \sum_{i=1}^n (M_i^2 - m_i^2)\,\Delta x_i \\ &= \sum_{i=1}^n (M_i - m_i)(M_i + m_i)\,\Delta x_i \\ &\le 2M \left\{\sum_{i=1}^n (M_i - m_i)\,\Delta x_i\right\} \\ &= 2M\,\{U(P, |f|) - L(P, |f|)\} \\ &< 2M\,\frac{\varepsilon}{2M} = \varepsilon\,. \end{aligned}$$

$$\Rightarrow \qquad f^2 \in \mathbf{R}\,[a, b].$$

Corollary: If f_1 and f_2 are both integrable on $[a, b]$, then $f_1 f_2$ is also integrable on $[a, b]$.

Since f_1, f_2 are integrable on $[a, b]$, therefore, f_1^2, f_2^2 and $(f_1 + f_2)^2$ are all integrable on $[a, b]$. Also

$$f_1 f_2 = \frac{1}{2}\{(f_1 + f_2)^2 - f_1^2 - f_2^2\}$$

therefore, $f_1 f_2 \in \mathbf{R}\,[a, b]$.

9.7 THE INTEGRAL AS A LIMIT OF SUMS (*Riemann Sums*)

Earlier, we arrived at the integral of a function via the upper and the lower sums. The numbers M_i, m_i which appear in these sums are not necessarily the values of the function f (they are values of f if f is continuous). We shall now show that $\int f\,dx$ can also be considered as the limit of a sequence of sums in which M_i and m_i are replaced by the values of f.

Corresponding to a partition P of $[a, b]$, let us choose points $t_1, t_2, \ldots, t_n$ such that $x_{i-1} \le t_i \le x_i$ $(i = 1, 2, \ldots, n)$ and let us consider the sum

$$S(P, f) = \sum_{i=1}^{n} f(t_i)\,\Delta x_i$$

The sum $S(P, f)$ is called *a Riemann sum of f over* $[a, b]$ *relative to* P.

It may be noted that t_i are arbitrary and that t_i can be any point whatsoever of Δx_i

We say that $S(P, f)$ converges to A as $\mu(P) \to 0$, *i.e.*,

$$\lim_{\mu(P)\to 0} S(P, f) = A$$

if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| S(P, f) - A \right| < \varepsilon$$

for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with norm $\mu(P) < \delta$ and for every choice of points t_i in $[x_{i-1}, x_i]$.

Definition 2. (*Second definition of integrability*). *A function f is said to be integrable on* $[a, b]$ *if* $\lim S(P, f)$ *exists as* $\mu(P) \to 0$, *and then*

$$\lim_{\mu(P)\to 0} S(P, f) = \int_a^b f\,dx\,.$$

Note. Since $\mu(P) \to 0$ when $n \to \infty$, therefore $\lim_{\mu(P)\to 0}$ can be replaced by $\lim_{n\to\infty}$ in the above definition.

We have, thus given two definitions of integrability. The equivalence of the two will now be established.

Def. 1 ⇒ Def. 2. Let a bounded function f be integrable according to the former definition, so that

$$\int_a^b f\,dx = \bar{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx$$

Let ε be any positive number.

By Darboux's Theorem, there exists $\delta > 0$ such that for every partition P with norm $\mu(P) < \delta$,

$$U(P,f) < \bar{\int_a^b} f\,dx + \varepsilon = \int_a^b f\,dx + \varepsilon \qquad \ldots(1)$$

and

$$L(P,f) > \bar{\int_a^b} f\,dx - \varepsilon = \int_a^b f\,dx - \varepsilon \qquad \ldots(2)$$

If t_i is any point of Δx_i, we have

$$L(P,f) \le \sum_{i=1}^{n} f(t_i)\,\Delta x_i \le U(P,f) \qquad \ldots(3)$$

From (1), (2), (3) we deduce that for any $\varepsilon > 0\ \exists\ \delta > 0$ such that for every partition P with norm $\mu(P) < \delta$,

$$\int_a^b f\,dx - \varepsilon < \sum_{i=1}^{n} f(t_i)\,\Delta x_i < \int_a^b f\,dx + \varepsilon$$

$$\Rightarrow \qquad \left| \sum_{i=1}^{n} f(t_i)\,\Delta x_i - \int_a^b f\,dx \right| < \varepsilon$$

Thus the function is integrable according to Def. 2 also.

Remark. Thus $f \in \mathbf{R} \Rightarrow \lim S(P,f)$ exists.

Def. 2 ⇒ Def. 1. Let a function f be integrable according to the second definition, *i.e.*,

$$\lim_{\mu(P)\to 0} S(P,f) \text{ exists}$$

In other words, to every number $\varepsilon > 0$, there exists $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ with norm $\mu(P) < \delta$ and for every choice of points t_i in Δx_i, $\exists$ a number A, such that

$$\left| \sum_{i=1}^{n} f(t_i)\,\Delta x_i - A \right| < \varepsilon\ .$$

It will first be shown that f is bounded.

Let, if possible, f be not bounded.

Taking $\varepsilon = 1$, there exists a partition P such that for every choice of t_i in Δx_i

$$\left| \sum f(t_i)\,\Delta x_i - A \right| < 1$$

$$\Rightarrow \qquad \left| \sum f(t_i)\,\Delta x_i \right| < |A| + 1$$

As f is not bounded in $[a, b]$ it must also be so in at least one subinterval, say Δx_m.

Let us take $t_i = x_i$ when $i \neq m$ so that every t_i except t_m is fixed and accordingly every term of the sum $\sum f(t_i)\,\Delta x_i$ except the term $f(t_m)\,\Delta x_m$ is also fixed. Since f is not bounded in Δx_m, we can choose a point t_m in Δx_m, such that

$$\left|\sum f(t_i)\Delta x_i\right| > |A| + 1$$

and thus we arrive at a contradiction.

Hence, the function f is bounded on $[a, b]$.

Now, let ε be any positive number. Thus there exists $\delta > 0$ such that for all partitions P with $\mu(P) < \delta$, we have

$$A - \frac{1}{2}\varepsilon < S(P, f) < A + \frac{1}{2}\varepsilon \qquad ...(4)$$

We choose one such P. If we let the points t_i range over the intervals Δx_i and take the l.u.b. and the g.l.b. of the numbers $S(P, f)$ obtained in this way, (4) yields

$$A - \frac{1}{2}\varepsilon < L(P, f) \leq U(P, f) < A + \frac{1}{2}\varepsilon \qquad ...(5)$$

$$\Rightarrow U(P, f) - L(P, f) < \varepsilon$$

Also

$$L(P, f) \leq \underline{\int}_a^b f\,dx \leq \bar{\int}_a^b f\,dx \leq U(P, f)$$

$$\therefore \qquad \bar{\int}_a^b f\,dx - \underline{\int}_a^b f\,dx \leq U(P, f) - L(P, f) < \varepsilon$$

so that a non-negative number is less than every positive number.

$$\Rightarrow \bar{\int}_a^b f\,dx - \underline{\int}_a^b f\,dx = 0$$

or

$$\bar{\int}_a^b f\,dx = \underline{\int}_a^b f\,dx$$

so that the function is integrable.

9.7.1 Example 9.4. Show that $\int_1^2 f\,dx = \frac{11}{2}$, where $f(x) = 3x + 1$.

Solution. Let $P = \{1 = x_0, x_1, x_2, \ldots, x_n = 2\}$ be a partition which divides $[1, 2]$ into n equal sub-intervals, each of length $\frac{2-1}{n} = \frac{1}{n}$, so that

$$\mu(P) = \frac{1}{n} \to 0 \text{ as } n \to \infty$$

$$x_i = 1 + \frac{i}{n},\ i = 1, 2, \ldots, n$$

$$\Delta x_i = \frac{1}{n},\ i = 1, 2, \ldots, n$$

$$\sum_{i=1}^{n} \Delta x_i = n.\frac{1}{n} = 1$$

Let $t_i = x_i$, when $i = 1, 2, \ldots, n$.

$$\therefore \quad S(P, f) = \sum_{i=1}^{n} f(t_i)\,\Delta x_i = \sum_{i=1}^{n} f(x_i)\,\Delta x_i = \sum_{i=1}^{n} (3x_i + 1)\,\Delta x_i$$

$$= \sum_{i=1}^{n} \left\{ 3\left(1 + \frac{i}{n}\right) + 1 \right\} \Delta x_i = 4 \sum_{i=1}^{n} \Delta x_i + \frac{3}{n^2} \sum_{i=1}^{n} i$$

$$= 4 + \frac{3}{n^2} \frac{n(n+1)}{2} = \frac{11}{2} + \frac{3}{2n}$$

Proceeding to limits when $\mu(P) \to 0$,

$$\lim_{\mu(P) \to 0} S(P, f) = \frac{11}{2}$$

Since the limit exists, the function is integrable and

$$\int_1^2 f\,dx = \lim S(P, f) = \frac{11}{2}$$

Example 9.5. Compute $\int_{-1}^{1} f\,dx$, where $f(x) = |x|$.

Solution. The function f is bounded and continuous on $[-1, 1]$, and

$$f(x) = \begin{cases} -x, & x \le 0 \\ x, & x \ge 0 \end{cases}$$

Let partition $P = \{-1 = x_0, x_1, \ldots, x_n = 0 = y_0, y_1, \ldots, y_n = 1\}$ divide $[-1, 1]$ into $2n$ equal subintervals each of length $1/n$, so that

$$\mu(P) = 1/n \to 0 \text{ as } n \to \infty;$$

$$x_i = -1 + i/n,\ \ i = 1, 2, \ldots, n;$$

$$y_i = i/n,\ \ i = 1, 2, \ldots, n;$$

$$\Delta x_i = 1/n = \Delta y_i$$

$$\sum_{i=1}^{n} \Delta x_i = 1 = \sum_{i=1}^{n} \Delta y_i$$

Let $t_i \in \Delta x_i$ and $t_i' \in \Delta y_i$
and let $t_i = x_i$ and $t_i' = y_i$ $\Big\}\ i = 1, 2, \ldots, n$

$$S(P, f) = \sum_{i=1}^{n} f(t_i)\,\Delta x_i + \sum_{i=1}^{n} f(t_i')\,\Delta y_i$$

$$= \sum_i (-x_i)\,\Delta x_i + \sum_i y_i\,\Delta y_i$$

$$= \sum_i \left(1 - \frac{i}{n}\right)\Delta x_i + \sum_i \frac{i}{n}\Delta y_i$$

$$= \sum_{i=1}^{n} \Delta x_i - \frac{1}{n^2}\sum_{i=1}^{n} i + \frac{i}{n^2}\sum_{i=1}^{n} i \;=\; 1$$

$\therefore \qquad \lim_{\mu(P)\to 0} S(P, f) = 1$

and since the limit exists, the function is integrable and

$$\int_{-1}^{1} |x|\,dx = \lim S(P, f) = 1\,.$$

9.7.2 Some Applications

I. *If functions* f, f_1, f_2, *where* $f = f_1 \pm f_2$ *are bounded and integrable on* $[a, b]$, *then*

$$\int_a^b f\,dx = \int_a^b f_1\,dx \pm \int_a^b f_2\,dx$$

Let ε be any positive number and $f = f_1 + f_2$.

Since f_1, f_2 are integrable therefore $\exists\ \delta > 0$ such that for every partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ with norm $\mu(P) < \delta$ and for every choice of points t_i in Δx_i.

$$\left.\begin{aligned} \left|\sum_i f_1(t_i)\,\Delta x_i - \int_a^b f_1\,dx\right| &< \frac{1}{2}\,\varepsilon \\ \left|\sum_i f_2(t_i)\,\Delta x_i - \int_a^b f_2\,dx\right| &< \frac{1}{2}\,\varepsilon \end{aligned}\right\}$$

$$\Rightarrow \left|\sum \{(f_1 + f_2)(t_i)\}\,\Delta x_i - \left\{\int_a^b f_1\,dx + \int_a^b f_2\,dx\right\}\right| < \varepsilon$$

$\therefore \qquad \int_a^b f\,dx = \int_a^b (f_1 + f_2)\,dx = \int_a^b f_1\,dx + \int_a^b f_2\,dx$

The case of $f = f_1 - f_2$ may be discussed similarly.

Corollary. If $f_1 \in \mathbf{R}$ and $f_2 \in \mathbf{R}$ over $[a, b]$, and c_1, c_2 any two constants, then $c_1 f_1 + c_2 f_2 \in \mathbf{R}$ $[a, b]$ and

$$\int_a^b (c_1 f_1 + c_2 f_2)\, dx = c_1 \int_a^b f_1\, dx + c_2 \int_a^b f_2\, dx$$

Let $f = c_1 f_1 + c_2 f_2$.

For any partition P, we can write

$$S(P, f) = \sum_i f(t_i)\, \Delta x_i = c_1 \sum_i f_1(t_i)\, \Delta x_i + c_2 \sum_i f_2(t_i)\, \Delta x_i$$

$$= c_1 S(P, f_1) + c_2 S(P, f_2)$$

Since f_1, f_2 are integrable, for $\varepsilon > 0$, we can choose $\delta > 0$ such that for all partition P with $\mu(P) < \delta$, we have

$$\left| S(P, f_1) - \int_a^b f_1\, dx \right| < \varepsilon$$

and

$$\left| S(P, f_2) - \int_a^b f_2\, dx \right| < \varepsilon$$

$$\therefore \quad \left| S(P, f) - c_1 \int_a^b f_1\, dx - c_2 \int_a^b f_2\, dx \right| < c_1 \varepsilon + c_2 \varepsilon$$

$$\Rightarrow \quad \lim_{\mu(P) \to 0} S(P, f) \text{ exists and equals } c_1 \int_a^b f_1\, dx + c_2 \int_a^b f_2\, dx$$

Hence $(c_1 f_1 + c_2 f_2) \in \mathbf{R}$, and

$$\int_a^b (c_1 f_1 + c_2 f_2)\, dx = c_1 \int_a^b f_1\, dx + c_2 \int_a^b f_2\, dx .$$

II. *If a function f is bounded and integrable on each of the intervals* [*a*, *c*], [*c*, *b*], [*a*, *b*] *where c is a point of* [*a*, *b*], *then*

$$\int_a^b f\, dx = \int_a^c f\, dx + \int_c^b f\, dx$$

Let $\varepsilon > 0$ be given.

As f is integrable on each of the intervals [*a*, *c*], [*c*, *b*] and [*a*, *b*], there exists $\delta > 0$ such that for every partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$ containing the point c, with norm $\mu(P) < \delta$ and for every choice of points t_i in Δx_i,

$$\left| \sum_{[a,c]} f(t_i)\, \Delta x_i - \int_a^c f\, dx \right| < \frac{1}{3} \varepsilon$$

$$\left| \sum_{[c,b]} f(t_i)\,\Delta x_i - \int_c^b f\,dx \right| < \frac{1}{3}\varepsilon$$

$$\left| \sum_{[a,b]} f(t_i)\,\Delta x_i - \int_a^b f\,dx \right| < \frac{1}{3}\varepsilon$$

But

$$\sum_{[a,c]} f(t_i)\,\Delta x_i + \sum_{[c,b]} f(t_i)\,\Delta x_i = \sum_{[a,b]} f(t_i)\,\Delta x_i$$

Therefore, we deduce that

$$\left| \int_a^b f\,dx - \int_a^c f\,dx - \int_c^b f\,dx \right| < \varepsilon$$

$$\Rightarrow \qquad \int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx$$

III. *A function f is integrable over* $[a, b]$ *iff for* $\varepsilon > 0, \exists\, \delta > 0$ *such that if* P, P' *are any two partitions of* $[a, b]$ *with mesh less than* δ, *then*

$$\left| S(P,f) - S(P',f) \right| < \varepsilon\,.$$

[This is the analog for Riemann sums of the Cauchy property of sequences.]

First, let $f \in \mathbf{R}\,[a, b]$, and $\int_a^b f\,dx = I$

$\therefore$ for $\varepsilon > 0, \exists\, \delta > 0$ such that for all partitions P, P' of $[a, b]$ with mesh less than δ and all positions of t_i in Δx_i,

$$\left| S(P,f) - I \right| < \frac{1}{2}\,\varepsilon \qquad \ldots(1)$$

$$\left| S(P',f) - I \right| < \frac{1}{2}\,\varepsilon \qquad \ldots(2)$$

$$\Rightarrow \qquad \left| S(P,f) - S(P',f) \right| \le \left| S(P,f) - I \right| + \left| S(P',f) - I \right|$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\,\varepsilon = \varepsilon$$

Conversely, for $\varepsilon > 0\ \exists\, \delta_1 > 0$ such that for any partitions P, P' with norm less than δ_1 we have

$$\left| S(P,f) - S(P',f) \right| < \frac{1}{2}\,\varepsilon\,. \qquad \ldots(3)$$

We know that for a given partition P' and every choice of t_i in Δx_i, $S(P',f)$ is bounded by $L(P',f)$ and $U(P',f)$, which for all partitions of $[a, b]$ are, in turn, bounded by $m\,(b-a)$ and $M\,(b-a)$, where m, M are bounds of f.

Thus the sequence $\{S(P',f)\}$ of Riemann sums is bounded. As every bounded sequence has a limit point, let the sequence have a limit point I, mesh so that

$$\lim_{\mu(P')\to 0} S(P',f) = I$$

$\therefore$ for $\varepsilon > 0, \exists\, \delta_2 > 0$ such that for partition P' with mesh

$$\mu(P') < \delta_2 \ ,$$

$$\left| S(P', f) - I \right| < \frac{1}{2}\,\varepsilon \qquad \ldots(4)$$

Let $\delta = \min(\delta_1, \delta_2)$.

$$\therefore \qquad \left| S(P, f) - I \right| \le \left| S(P, f) - S(P', f) \right| + \left| S(P', f) - I \right|$$

$$< \frac{1}{2}\,\varepsilon + \frac{1}{2}\,\varepsilon = \varepsilon, \quad \text{for } \mu(P) < \delta$$

$$\Rightarrow \qquad \lim_{\mu(P)\to 0} S(P, f) = I$$

Thus $\lim_{\mu(P)\to 0} S(P, f)$ exist and hence the function f is integrable.

9.8 SOME INTEGRABLE FUNCTIONS

Theorem 9.12. *Every continuous function is integrable.*

We shall prove that a function f which is continuous on $[a, b]$ is also integrable on $[a, b]$.

Let $\varepsilon > 0$ be given.

Let us choose a positive number η, such that

$$\eta(b - a) < \varepsilon.$$

Since f is continuous on the closed interval $[a, b]$, therefore, it is bounded and is *uniformly continuous* on $[a, b]$, which implies that there exists $\delta > 0$, such that

$$\left| f(t_1) - f(t_2) \right| < \eta, \text{ if } \left| t_1 - t_2 \right| < \delta, \text{ and } t_1, t_2 \in [a, b] \qquad \ldots(1)$$

We, now, choose a partition P with norm $\mu(P) < \delta$.

Then by (1), we have

$$M_i - m_i \le \eta \quad (i = 1, 2, \ldots, n)$$

Hence

$$U(P, f) - L(P, f) = \sum_{i=1}^{n} (M_i - m_i)\,\Delta x_i$$

$$\le \eta \sum_{i=1}^{n} \Delta x_i$$

$$= \eta(b - a) < \varepsilon \qquad \ldots(2)$$

Thus, f is integrable. [§ 9.5 Theorem 9.3].

Corollary. If a function f is continuous, then to every $\varepsilon > 0$ there corresponds $\delta > 0$ such that

$$\left| \sum_{i=1}^{n} f(t_i)\,\Delta x_i - \int_a^b f\,dx \right| < \varepsilon,$$

for every partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a, b]$ *with* $\mu(P) < \delta$, and for every choice of points t_i in $[x_{i-1}, x_i]$.

Follows from (2) above, since the two numbers $\sum f(t_i)\,\Delta x_i$ and $\int f\,dx$ lie between $U(P, f)$ and $L(P, f)$.

Remark. For continuous functions, $\lim_{\mu(P)\to 0} \sum f(t_i)\,\Delta x_i$ exists and equals $\int_a^b f\,dx$.

Note. Continuity is a sufficient condition for integrability. It is not a necessary condition. Functions exist which are integrable but not continuous. See Examples 9.6, 9.7 and 9.8. Many more may be constructed.

Theorem 9.13. *If a function f is monotonic on $[a, b]$, then it is integrable on $[a, b]$.*

We shall prove the theorem when f is monotonic increasing (the proof for the other case is analogous).

Clearly f is bounded.

Let $\varepsilon > 0$ be given.

Let us choose a number $\eta < \dfrac{\varepsilon}{f(b) - f(a) + 1}$.

We, now, choose a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$, with norm $\mu(P) < \eta$.

Since f is monotonic increasing, therefore,

$$M_i = f(x_i),\ m_i = f(x_{i-1}), \quad (i = 1, 2, \ldots, n)$$

Hence

$$\begin{aligned} U(P,f) - L(P,f) &= \sum_{i=1}^{n} (M_i - m_i)\,\Delta x_i \\ &= \sum_{i=1}^{n} \{f(x_i) - f(x_{i-1})\}\,\Delta x_i \\ &< \eta \sum_{i=1}^{n} \{f(x_i) - f(x_{i-1})\} \\ &< \frac{\varepsilon}{f(b) - f(a) + 1} \cdot \{f(b) - f(a)\} \\ &< \varepsilon. \end{aligned}$$

Hence, f is integrable.

Note. $f(b) - f(a) + 1$ have been taken to cover the case when $f(a) = f(b)$.

Theorem 9.14. *A bounded function f, having a finite number of points of discontinuity on $[a, b]$, is integrable on $[a, b]$.*

Let M, m be the bounds of f.

$\varepsilon > 0$ be a given number.

Let there be p points of discontinuity of f on $[a, b]$.

We now consider a partition P of $[a, b]$, such that all the points of discontinuity get enclosed in p non-overlapping sub-intervals, the sum of whose lengths $< \frac{1}{2}\varepsilon \Big/ (M - m)$. The oscillation of f in each of these sub-intervals being $\leq (M - m)$, their total contribution to the difference $\{U(P,f) - L(P,f)\}$ is less than

$$\frac{\varepsilon}{2(M-m)}(M-m)=\frac{1}{2}\varepsilon.$$

The function f is continuous in the remaining portion of $[a, b]$, *i.e.*, in the $(p+1)$ sub-intervals of $[a, b]$ excluding the sub-intervals considered above.

As in Theorem 9.12, the contribution to the difference $\{U(P,f) - L(P,f)\}$ from each of these $(p+1)$ sub-intervals can be made $< \dfrac{\varepsilon}{2(p+1)}$, so that the total contribution to $\{U(P,f) - L(P,f)\}$ by these $(p+1)$ sub-intervals is less than $\dfrac{\varepsilon}{2(p+1)}(p+1)=\dfrac{1}{2}\varepsilon$.

Thus, for the partition P of $[a, b]$,

$$U(P,f) - L(P,f) < \varepsilon.$$

Hence the function f is integrable.

Theorem 9.15. *A bounded function f is integrable on* $[a, b]$, *if the set of its points of discontinuity has only a* **finite number of limit points**.

Let the set of points of discontinuity of f have a finite number p of limit points. Let M, m be the bounds of f.

The limit points may be enclosed in p non-overlapping sub-intervals of $[a, b]$, the sum of whose lengths $\leq \dfrac{1}{2}\varepsilon \Big/ (M-m)$. So that their total contribution to $\{U(P,f) - L(P,f)\}$ is $< \dfrac{1}{2}\varepsilon$.

Only a finite number of points of discontinuity of f can be outside these sub-intervals, *i.e.*, the function f has a finite number of points of discontinuity on $[a, b]$ excluding the p sub-intervals enclosing the limit points. Therefore, as in Theorem 9.14, the total contribution to $\{U(P,f) - L(P,f)\}$ from these portions of $[a, b]$ can be made $< \dfrac{1}{2}\varepsilon$.

Thus for such a partition P of $[a, b]$,

$$U(P,f) - L(P,f) < \varepsilon.$$

Hence, the function f is integrable on $[a, b]$.

Example 9.6. A function f is defined on $[-1, 1]$ as follows:

$$f(x)=\begin{cases} k, & \text{positive constant when } x \neq 0 \\ 0, & \text{when } x = 0. \end{cases}$$

Show that f is integrable on $[-1, 1]$ and that the value of the integral is $2k$.

Solution. The function has only one point of discontinuity, 0, and is therefore integrable (Theorem 9.14). Proceeding as in § 9.2 Example 9.1, it may be shown that the value of the integral is $2k$.

Example 9.7. A function f is defined on $[0, 1]$ as follows:

$$f(x)=\begin{cases} 0, & \text{when } x \text{ is irrational or zero, and} \\ 1/q, & \text{when } x \text{ is any non-zero rational number } p/q \text{ with least positive integers } p \text{ and } q. \end{cases}$$

Show that f is integrable on $[0, 1]$ and the value of the integral is zero.

Solution. The function f is bounded with bounds 0 and 1.

Let ε be any positive number.

There exists a largest integer $q \in \mathbf{N}$ such that $1/q > \frac{1}{2}\varepsilon$ or $q < 2/\varepsilon$, so that there are only a finite number of points p/q for which $1/q > \frac{1}{2}\varepsilon$ Let us call such points as exceptional points.

Thus at those rational points which are exceptional points, f has a value $1/q > \frac{1}{2}\varepsilon$, while at the other rational points, the value of f is $1/q > \frac{1}{2}\varepsilon$. The function is zero at the irrational points.

Also, every interval contains rational as well as irrational points.

Thus the oscillation of f in any interval which includes no exceptional points is less than $\frac{1}{2}\varepsilon$ and that in an interval which includes the exceptional points it is at the most equal to 1.

Let us consider a partition P of [0, 1] so as to enclose the exceptional points (finite in number) in subintervals, the sum of whose lengths is less than $\frac{1}{2}\varepsilon$. Thus the contribution to $[U(P, f) - L(P, f)]$ made by these is less than $\frac{1}{2}\varepsilon$.

The contribution to $[U(P, f) - L(P, f)]$ by the remaining portion of [0, 1] is evidently less than $\frac{1}{2}\varepsilon$.

Hence, $U(P, f) - L(P, f) < \varepsilon$, so that the function f is integrable.

Example 9.8. Show that the function f defined as follows:

$$f(x) = \frac{1}{2^n}, \text{ when } \frac{1}{2^{n+1}} < x \le \frac{1}{2^n}, (n = 0, 1, 2, \ldots),$$

$$f(0) = 0,$$

is integrable on [0, 1], although it has an infinite number of points of discontinuity.

Solution. Now

$$\begin{aligned} f(x) &= 1, \text{ when } \frac{1}{2} < x \le \frac{1}{1} \\ &= \frac{1}{2}, \text{ when } \frac{1}{2^2} < x \le \frac{1}{2} \\ &= \frac{1}{2^2}, \text{ when } \frac{1}{2^3} < x \le \frac{1}{2^2} \\ &\quad \cdots \quad \cdots \quad \cdots \\ &\quad \cdots \quad \cdots \quad \cdots \\ &= \frac{1}{2^{n-1}}, \text{ when } \frac{1}{2^n} < x \le \frac{1}{2^{n-1}} \\ &\quad \cdots \quad \cdots \quad \cdots \\ &= 0, \text{ when } x = 0. \end{aligned}$$

Thus we notice that f is bounded and monotonic increasing on [0, 1].

Hence, f is integrable [Theorem 9.13].

Aliter: f is continuous on [0, 1], except at the set of points

$$0, \frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3 \ldots.$$

which have only one limit point, 0, and hence f is integrable.

9.9 INTEGRATION AND DIFFERENTIATION (*The Primitive*)

We shall first show that integration and differentiation are in a certain sense, inverse operations, then define the primitive of a function and go on to prove a theorem which is usually called the fundamental theorem of calculus.

Theorem 9.16. *If a function f is bounded and integrable on* $[a, b]$, *then the function F defined as*

$$F(x) = \int_a^x f(t)\,dt,\ a \le x \le b,$$

is continuous on $[a, b]$, *and furthermore, if f is continuous at a point c of* $[a, b]$, *then F is derivable at c and*

$$F'(c) = f(c).$$

Since f is bounded therefore $\exists$ a number $K > 0$, such that

$$|f(x)| \le K, \quad \forall\ x \in [a, b].$$

If x_1, x_2 are two points of $[a, b]$ such that $a \le x_1 < x_2 \le b$, then

$$|F(x_2) - F(x_1)| = \left| \int_{x_1}^{x_2} f(t)\,dt \right|$$

$$\le K(x_2 - x_1) \qquad (\S\ 1.3 \text{ Deduction } 3)$$

Thus for a given $\varepsilon > 0$, we see that

$$|F(x_2) - F(x_1)| < \varepsilon, \quad \text{if } |x_2 - x_1| < \varepsilon/K.$$

Hence the function F is continuous (in fact uniformly) on $[a, b]$.

Let f be continuous at a point c of $[a, b]$, so that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon, \quad \text{for } |x - c| < \delta$$

Let $c - \delta < s \le c \le t < c + \delta$

$$\therefore \qquad \left| \frac{F(t) - F(s)}{t - s} - f(c) \right| = \left| \frac{1}{t - s} \int_s^t \{f(x) - f(c)\}\,dx \right|$$

$$\le \frac{1}{t - s} \int_s^t |f(x) - f(c)|\,dx < \varepsilon.$$

$$\Rightarrow \qquad F'(c) = f(c),$$

i.e., continuity of f at any point of $[a, b]$ implies derivability of $\boldsymbol{F}$ at that point.

Note. As c is any point of $[a, b]$, we have for all $x \in [a, b]$,

$$F'(t) = f(t) \Rightarrow F' = f$$

i.e., continuity of f on $[a, b]$ implies derivability of F on $[a, b]$.

This theorem is sometimes referred to as the *First Fundamental Theorem of Integral Calculus*.

Definition. A derivable function F, if it exists such that its derivative F' is equal to a given function f, is called a **primitive** *of f.*

The above theorem shows that a sufficient condition for a function to admit of a primitive is that it is continuous. Thus every continuous function f possesses a primitive F, where

$$F(x) = \int_a^x f(t)\, dt$$

Remark. We shall now show, with the help of an example, that continuity of a function is not a necessary condition for the existence of a primitive, in other words, "*functions possessing primitives are not necessarily continuous*".

Consider the function f on $[0, 1]$, where

$$f(x) = \begin{cases} 2x, \sin(1/x) - \cos(1/x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

It has a primitive F, where

$$F(x) = \begin{cases} x^2 \sin 1/x, & \text{if } x \neq 0. \\ 0, & \text{if } x = 0. \end{cases}$$

Clearly $F'(x) = f(x)$ but $f(x)$ is not continuous at $x = 0$, *i.e.*, $f(x)$ is not continuous on $[0, 1]$ [Example 6.4, Ch. 6].

In fact, all this amounts to saying that the derivative of a function is not necessarily continuous.

9.10 THE FUNDAMENTAL THEOREM OF CALCULUS

Theorem 9.17. *A function f is bounded and integrable on* $[a, b]$, *and there exists a function F such that* $F' = f$ *on* $[a, b]$, *then*

$$\int_a^b f\, dx = F(b) - F(a).$$

Since the function $F' = f$ is bounded and integrable, therefore for every given $\varepsilon > 0, \exists\, \delta > 0$ such that for every partition $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$, with norm $\mu(P) < \delta$,

$$\left.\begin{aligned} \sum_{i=1}^{n} f(t_i)\, \Delta x_i - \int_a^b f\, dx \,| < \varepsilon \\ \lim_{(P)\to 0} \sum_{i=1}^{n} f(t_i)\, \Delta x_i = \int_a^b f\, dx, \end{aligned}\right\} \quad \text{or} \quad \ldots(1)$$

for every choice of points t_i in Δx_i.

Since, we have freedom in the selection of points t_i in Δx_i, we choose them in a particular way as follows:

By Lagrange's Mean Value Theorem, we have

$$F(x_i) - F(x_{i-1}) = F'(t_i)\,\Delta x_i \quad (i = 1, 2, \ldots, n)$$
$$= f(t_i)\,\Delta x_i.$$

$$\Rightarrow \quad \sum_{i=1}^{n} f(t_i)\,\Delta x_i = \sum_{i=1}^{n} \{F(x_i) - F(x_{i-1})\}$$
$$= F(b) - F(a).$$

Hence from (1),

$$\int_a^b f\,dx = F(b) - F(a)$$

It is sometimes referred to as *The Second Fundamental Theorem of Integral Calculus*.

9.10.1 Solved Examples

Example 9.9. Show that $\int_0^1 f\,dx = \frac{2}{3}$, where f is the integrable function in Example 9.8.

Solution. The function f is integrable

$$\therefore \quad \int_{1/2^n}^{1} f\,dx = \int_{1/2}^{1} f\,dx + \int_{1/2^2}^{1/2} f\,dx + \int_{1/2^3}^{1/2^2} f\,dx + \ldots + \int_{1/2^n}^{1/2^{n-1}} f\,dx$$

$$= \int_{1/2}^{1} dx + \frac{1}{2}\int_{1/2^2}^{1/2} dx + \frac{1}{2^2}\int_{1/2^3}^{1/2^2} dx + \ldots + \frac{1}{2^{n-1}}\int_{1/2^n}^{1/2^{n-1}} dx$$

$$= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2^2}\right) + \frac{1}{2^2}\left(\frac{1}{2^2} - \frac{1}{2^3}\right) + \ldots + \frac{1}{2^{n-1}}\left(\frac{1}{2^{n-1}} - \frac{1}{2^n}\right)$$

$$= \frac{1}{2}\left\{1 + \frac{1}{2^2} + \left(\frac{1}{2^2}\right)^2 + \left(\frac{1}{2^2}\right)^3 + \ldots + \left(\frac{1}{2^2}\right)^{n-1}\right\}$$

$$= \frac{1}{2}\cdot\frac{1 - \frac{1}{4''}}{1 - \frac{1}{4}} = \frac{2}{3}\left(1 - \frac{1}{4''}\right)$$

Proceeding to limits when $n \to \infty$, we get

$$\int_0^1 f\,dx = \frac{2}{3}.$$

Example 9.10. Show that the function $[x]$, where $[x]$ denotes the greatest integer not greater than x, is integrable in [0, 3], and

$$\int_0^3 [x]dx = 3.$$

Solution. The function is bounded and has only three points of finite discontinuity at 1, 2, 3.

Let ε be a given positive number.

Consider a partition P where,

$$P = \{0 = x_0, x_1, \ldots, x_l, y_0, y_1, \ldots, y_m, z_0, z_1, \ldots, z_n = 3\}$$

where $y_0 = 1, \quad z_0 = 2.$

Now

$$\begin{aligned} U(P,f) &= \sum 0 \cdot \Delta x_i + 1 \cdot (y_0 - x_l) + \sum 1 \cdot \Delta y_i + 2(z_0 - y_m) \\ &\quad + \sum 2 \cdot \Delta z_i + 3(z_n - z_{n-1}) \\ &= 1 + 2 + \{(y_0 - x_l) + (z_0 - y_m) + (z_n - z_{n-1})\} \end{aligned}$$

Let us select P such that

$$(y_0 - x_i) + (z_0 - y_m) + (z_n - z_{n-1}) < \varepsilon$$

$$\therefore \qquad U(P,f) < 3 + \varepsilon$$

Again,

$$L(P,f) = \sum 0 \cdot \Delta x_i + \sum 1 \cdot \Delta y_i + (z_0 - y_m) + \sum 2 \cdot \Delta z_i = 3$$

$$\therefore \qquad U(P,f) - L(P,f) < \varepsilon$$

so that the function is integrable, and therefore

$$\bar{\int_0^3} [x]\, dx = \underline{\int_0^3} [x]\, dx = 3.$$

Aliter: Since the function is bounded and has only three points of discontinuity therefore it is integrable, and

$$\begin{aligned} \int_0^3 [x]dx &= \int_0^1 [x]dx + \int_1^2 [x]dx + \int_2^3 [x]dx \\ &= \int_0^1 0\, dx + \int_1^2 1\, dx + \int_2^3 2\, dx = 3 \end{aligned}$$

1, 2 and 3 being respectively the points of discontinuity of the three integrals on the right.

Example 9.11. f is a non-negative continuous function on $[a, b]$, and

$$\int_a^b f\, dx = 0.$$ Prove that $f(x) = 0$, for all $x \in [a,b]$.

Solution. Suppose that, for some $c \in]a, b[$, $f(c) > 0$.

Then, for $\varepsilon = \frac{1}{2} f(c) > 0$, continuity of f at c implies that, there exists a $\delta > 0$ such that

$$f(x) > \frac{1}{2} f(c), \ \forall\ x \in]c-\delta, c+\delta[$$

Now

$$\int_a^b f(x)\,dx = \int_a^{c-\delta} f(x)\,dx + \int_{c-\delta}^{c+\delta} f(x)\,dx + \int_{c+\delta}^{b} f(x)\,dx$$

$$\geq \int_{c-\delta}^{c+\delta} f(x)\,dx \qquad (\because\ f(x) \geq 0, \ \forall\ x \in [a, b])$$

$$> \frac{1}{2} f(c) \int_{c-\delta}^{c+\delta} dx = \delta f(c) > 0.$$

which is a contradiction. Thus $f(x) = 0, \ \forall\ x \in]a, b[$.

Similarly, $f(a) \not> 0$, and $f(b) \not> 0$. Hence the result follows.

Example 9.12. Show that $\int_0^t \sin x\ dx = 1 - \cos t$.

Solution. The function $\sin x$ is bounded and continuous in any interval $[0, t]$ and is therefore integrable. [To be more specific, take $t \leq \pi/2$.]

Consider a partition $P = \left\{0, \frac{t}{n}, \frac{2t}{n}, \ldots, \frac{nt}{n}\right\}$ of $[0, t]$.

Now

$$U(P, \sin x) = \frac{t}{n}\left\{\sin\frac{t}{n} + \sin\frac{2t}{n} + \sin\frac{3t}{n} + \ldots + \sin\frac{nt}{n}\right\}$$

$$= \frac{t}{n}\left(\cos\frac{t}{2n} - \cos(n+1)\frac{t}{n}\right)\Big/\left(2\sin\frac{t}{2n}\right)$$

$$= \left\{\cos\frac{t}{2n} - \cos(n+1)\frac{t}{n}\right\}\Big/\left(\frac{\sin t/2n}{t/2n}\right)$$

and

$$L(P, \sin x) = \frac{t}{n}\left\{\sin\frac{t}{n} + \sin\frac{2t}{n} + \ldots + \sin n - 1\,\frac{t}{n}\right\}$$

$$= \left\{\cos\frac{t}{2n} - \cos\frac{nt}{n}\right\}\Big/\left(\frac{\sin t/2n}{t/2n}\right).$$

In the limit,

$$U(P, \sin x) = L(P, \sin x) = 1 - \cos t\ .$$

Hence

$$\int_0^t \sin x\ dx = 1 - \cos t$$

9.11 MEAN VALUE THEOREMS OF INTEGRAL CALCULUS

We have two Mean Value Theorems for derivatives. In the same way we have two mean value theorems for integrals as well.

9.11.1 First Mean Value Theorem

Theorem 9.18. *If a function f is continuous on* $[a, b]$, *then there exists a number* ξ *in* $[a, b]$ *such that*

$$\int_a^b f\,dx = f(\xi)(b-a).$$

f is continuous, therefore $f \in \mathbf{R}\ [a, b]$.

Let m, M be the infimum and supremum of f in $[a, b]$. Then as in § 1.1, we have

$$m(b-a) \le \int_a^b f\,dx \le M(b-a)$$

Hence, there is a number $\mu \in [m, M]$, such that

$$\int_a^b f\,dx = \mu\,(b-a)$$

Since f is continuous on $[a, b]$, it attains every value between its bounds m, M. Therefore, there exists a number $\xi \in [a, b]$ such that $f\ (\xi) = \mu$.

$$\therefore \qquad \int_a^b f\,dx = f(\xi)(b-a).$$

9.11.2 The Generalised First Mean Value Theorem

Theorem 9.19. *If f and g are integrable on* $[a, b]$ *and g keeps the same sign over* $[a, b]$, *then there exists a number* μ *lying between the bounds of f such that*

$$\int_a^b fg\,dx = \mu \int_a^b g\,dx.$$

Let g be positive over $[a, b]$.

If m, M are the bounds of f, we have for all $x \in |a, b|$,

$$m \le f(x) \le M$$

$$\Rightarrow \qquad mg\ (x) \le f(x)\,g(x) \le Mg(x)$$

Thus $$m\int_a^b g(x)\,dx \le \int_a^b f(x)\,g(x)\,dx \le M\int_a^b g(x)\,dx, \text{ when } b \ge a \qquad ...(1)$$

Let μ be a number lying between m and M.

$$\therefore \qquad \int_a^b fg\,dx = \int_a^b g\,dx \qquad ...(2)$$

Corollary. If in addition to the conditions of the above theorem, f is continuous on $[a, b]$, then $\exists$ a number $\xi \in [a, b]$, such that

$$\int_a^b fg\ dx = f(\xi) \int_a^b g\ dx \qquad ...(3)$$

Notes 1. If $g(x) \leq 0$, the sign of the inequality changes in (1) but (2) and (3) remain unchanged.

2. If $b \leq a$, the sign of inequality changes in (1) but (2) and (3) remain uneffected.

EXERCISE

1. Show that $\int_0^1 x^4\ dx = \frac{1}{5}$.
2. If f is continuous and non-negative on $[a, b]$. Then show that
$$\int_a^b f\ dx \geq 0.$$
3. If f is continuous and non-negative on $[a, b]$ and
$\int_a^b f dx = 0$, then prove that $f(x) = 0, \forall\ x \in [a, b]$.
4. If $f(x) = 0$ or 1 according as x is rational or non-rational, prove that f is not integrable on any interval.
5. Prove that the function f defined as
$$f(x) = \begin{cases} x, & \text{when } x \text{ is rational} \\ -x, & \text{when } x \text{ is not rational} \end{cases}$$
is not integrable over $[a, b]$; but $|f|$ is integrable.
6. Integrate on [0, 2] the function $f(x) = x[x]$, where $[x]$ denotes the greatest integer not greater than x.
7. f and g are two bounded functions on $[a, b]$ such that $f(x) = g(x)$ except for a finite number of points x in $[a, b]$. If g is integrable on $[a, b]$, then prove that f is so and in this case
$$\int_a^b f\ dx = \int_a^b g\ dx.$$
[*Hint*: $F(x) = f(x) - g(x) = 0$ except at a finite number of points of $[a, b]$, so that $F(x)$ has a finite number of points of discontinuity on $[a, b]$].
8. If $f(y, x) = 1 + 2x$, for y rational and $f(y, x) = 0$, for y irrational, find $F(y)$, where
$$F(y) = \int_0^1 f(y, x)\ dx.$$
9. Evaluate
$$\bar{\int}_0^2 f(x)\ dx$$

$$\text{where } f(x) = \begin{cases} 0, & \text{when } x = n/(n+1),\ (n+1)/n,\ (n = 1, 2, 3 \ldots) \\ 1, & \text{elsewhere.} \end{cases}$$

Is f integrable on $[0, 2]$? Examine for continuity the function f so defined at the point $x = 1$.

10. f is bounded and integrable on $[a, b]$: show that

$$\int_a^b [f(x)]^2 \, dx = 0$$

if and only if $f(c) = 0$, at every point, c, of continuity of f.

11. If f is integrable over $[a, b]$, under what conditions is $1/f$ integrable over $[a, b]$? State and prove a theorem about the integrability of $1/f$ over $[a, b]$.

12. A function f is defined on $[0, 1]$ as follows:

$$f(x) = \frac{1}{a^{r-1}} \text{ and } f(0) = 0, \text{ where } \frac{1}{a^r} < x \le \frac{1}{a^{r-1}}, \text{ for } r = 1, 2, 3, \ldots,$$

where a is an integer greater that 2.

Show that $\int_0^1 f \, dx$ exists and is equal to $\dfrac{a}{a+1}$.

13. A function f is defined on $[0, 1]$, for positive integral value of r such that

$$f(x) = (-1)^{r-1}; \text{ where } 1/(r+1) < x < 1/r,\ (r = 1, 2, 3, \ldots)$$
$$f(0) = 0.$$

Show that $\int_0^1 f \, dx = \log 4 - 1$.

9.12 INTEGRATION BY PARTS

Theorem 9.20 (*A particular case*). *If f and g are both differentiable on $[a, b]$ and if f' and g' are both integrable on $[a, b]$ then*

$$\int_a^b f(x)\, g'(x)\, dx = \Big[f(x) \cdot g(x)\Big]_a^b - \int_a^b g(x)\, f'(x)\, dx.$$

Since f and g are differentiable and hence continuous on $[a, b]$, therefore $f \in \mathbf{R}$, $g \in \mathbf{R}$ over $[a, b]$. Again, since f, g, f' and g' are all integrable over $[a, b]$, therefore $fg' \in \mathbf{R}$, $gf' \in \mathbf{R}$ over $[a, b]$.

Let $F(x) = f(x)\, g(x)$, $\forall\ x \in [a, b]$,

$$\therefore \qquad F'(x) = f(x)\, g'(x) + g(x)\, f'(x)$$

$$\therefore \qquad \int_a^b F'(x)\, dx = \int_a^b \{f(x)\, g'(x) + g(x)\, f'(x)\} dx$$

$$= \int_a^b f(x)\, g'(x)\, dx + \int_a^b g(x)\, f'(x)\, dx \qquad \ldots(1)$$

Also by Fundamental Theorem,

$$\int_a^b F'(x)\,dx = F(b) - F(a) = \Big[f(x)\,g(x)\Big]_a^b \qquad ...(2)$$

From (1) and (2), we get

$$\int_a^b f(x)\,g'(x)\,dx = \Big[f(x)\,g(x)\Big]_a^b - \int_a^b g(x)\,f'(x)\,dx$$

9.13 CHANGE OF VARIABLE IN AN INTEGRAL

Theorem 9.21. *If* (*i*) $f \in \mathbf{R}\,[a, b]$, (*ii*) ϕ *is a derivable, strictly monotonic function on* $[\alpha, \beta]$, *where* $a = \phi(\alpha), b = \phi(\beta)$, *and* (*iii*) $g' \in \mathbf{R}\,[\alpha, \beta]$, *then*

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\phi(y))\,\phi'(y)\,dy.$$

$$\left[\text{Change of variable in } \int_a^b f(x)\,dx \text{ by putting } x = \phi(y).\right]$$

Let ϕ be strictly monotonic increasing on $[\alpha, \beta]$.

Since ϕ is strictly monotonic, it is invertible, *i.e.*,

$$x = \phi(y) \Rightarrow y = \phi^{-1}(x), \quad \forall\ x \in [a, b]$$

so that

$$\alpha = \phi^{-1}(a) \text{ and } \beta = \phi^{-1}(b)$$

Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be any partition of $[a, b]$ and

$Q = \{\alpha = y_0, y_1, y_2, \ldots, y_n = \beta\}$, $y_i = \phi^{-1}(x_i)$ the corresponding partition of $[\alpha, \beta]$.

By Lagrange's Mean Value Theorem,

$$\Delta x_i = \phi(y_i) - \phi(y_{i-1}) = \phi'(\eta_i)\,\Delta y_i, \quad \eta_i \in \Delta y_i \qquad ...(1)$$

Let

$$\xi_i = \phi(\eta_i), \text{ where } \xi_i \in \Delta x_i \qquad ...(2)$$

Now,

$$S(P, f) = \sum_{i=1}^n f(\xi_i)\,\Delta x_i = \sum_{i=1}^n f(\phi(\eta_i))\,\phi'(\eta_i)\,\Delta y_i \qquad ...(3)$$

Uniform continuity of ϕ implies that $\mu(Q) \to 0$ as $\mu(P) \to 0$

Also $f \in \mathbf{R}$ implies $\lim_{\mu(P)\to 0} S(P, f)$ exists, and therefore, the limit of the R.H.S. of (3) as $\mu(Q) \to 0$ also exists and equals the integral $\int_\alpha^\beta f(\phi(y))\,\phi'(y)\,dy$.

Hence letting $\mu(P) \to 0$ in (3), we get

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\phi(y))\phi'(y)\,dy$$

Remarks 1. If $\phi'(y) \neq 0$ for any $y \in [\alpha, \beta]$, then ϕ is strictly monotonic in $[\alpha, \beta]$. Hence the condition of strict monotonicity of ϕ in the statement of the theorem, may be replaced by

$$\phi'(y) \neq 0, \quad \forall\ y \in [\alpha, \beta]$$

2. The theorem holds even if $\phi'(y) = 0$ for a finite number of values of y. For, in that case, the interval $[\alpha, \beta]$ can be divided into a finite number of sub-intervals, in each of which ϕ is strictly monotonic. Repetition of the argument for each of the sub-intervals in turn and the addition of the results, gives the required result.

9.14 SECOND MEAN VALUE THEOREM

Theorem 9.22. *If f is monotonic and f, f' and g are all continuous in $[a, b]$, then there exists $\xi \in [a, b]$ such that*

$$\int_a^b f(x)g(x)\,dx = f(a)\int_a^\xi g(x)\,dx + f(b)\int_\xi^b g(x)\,dx.$$

Let $G(x) = \int_a^x g(t)\,dt$

Clearly $G(a) = 0$, and under the given conditions, $G(x)$ is differentiable and $G'(x) = g(x)$.

$$\therefore \qquad \int_a^b f(x)g(x)\,dx = \int_a^b f(x)G'(x)\,dx$$

$$= \left[f(x)G(x)\right]_a^b - \int_a^b G(x)f'(x)\,dx,$$

on integrating by parts.

Since $G(x)$, being continuous, is integrable and $f(x)$ is also continuous on $[a, b]$, therefore on using generalised First Mean Value Theorem, $\exists\ \xi \in [a, b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(b)G(b) - G(\xi)\int_a^b f'(x)\,dx$$

$$= f(b)G(b) - G(\xi)\{f(b) - f(a)\}$$

$$= f(b)\{G(b) - G(\xi)\} + f(a)\,G(\xi)$$

$$= f(b)\int_\xi^b g(x)\,dx + f(a)\int_a^\xi g(x)\,dx$$

Note. For proofs of the theorem under less stringent conditions, see Hobson, Functions of a real variable.

EXERCISE

1. If f is continuous and positive on $[a, b]$ then show that $\int_a^b f\ dx$ is also positive.
2. Can the number ξ of the first mean value theorem always belong to $]a, b[$? Find the function f on some closed interval which satisfies the conditions of this theorem but for which ξ must be an end point of the interval.
3. How far can the Lagrange's mean value theorem for derivatives be used to make the statement: If f is the derivative of some function on $[a, b]$, then there exists a number $\xi \in [a, b]$ such that

$$\int_a^b f\ dx = f(\xi)\,(b - a)\,.$$

   [*Hint*: In Lagrange's mean value theorem, $\xi \in\ ]a, b[$]

4. If f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)\ dt$, for all $x \in [a, b]$, apply (if possible) the Lagrange's mean value theorem for derivatives to F over $[a, b]$. State the resulting theorem for F. Also state the first mean value theorem for integrals to $\int_a^b f\ dx$, and compare the two statements.

   Also try to relate Cauchy mean value theorem with the generalised first mean value theorem for integrals.
5. If f and g are integrable and

$$\int_a^b f\ dx = \int_a^b g\ dx,$$

   then show that

$$f(\xi) = g(\xi), \text{ for some } \xi \in [a, b].$$

CHAPTER 10

The Riemann-Stieltjes Integral

Having discussed the Riemann theory of integration to the extent possible within the scope of the present discussion, we now pass on to a generalisation of the subject. As mentioned earlier many refinements and extensions of the theory exist but we shall study briefly—in fact very briefly—the extension due to Stieltjes, known as the theory of *Riemann-Stieltjes integration*. The most noteworthy of the extensions, the *Lebesgue theory* of integration will be however beyond the scope of the present discussion.

It may be stated once for all that, unless otherwise stated, all functions will be *real-valued and bounded* on the domain of definition. The *function* α *will always be monotonic increasing*.

10.1 DEFINITIONS AND EXISTENCE OF THE INTEGRAL

Let f and α be bounded function on $[a, b]$ and α be *monotonic increasing* on $[a, b]$, $b \geq a$.

Corresponding to any partition

$$P = \{a = x_0, x_1, ..., x_n = b\}, \text{ of } [a, b],$$

we write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}), \quad i = 1, 2, ..., n.$$

It is clear that $\Delta\alpha_i \geq 0$. As in § 1.1 Ch. 9, we define two sums,

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$$

$$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$

where m_i, M_i, are the bounds (infimum and supremum respectively) of f in Δx_i, respectively called the **Upper** and the **Lower** sums of f corresponding to the partition P.

If m, M are respectively the lower and the upper bounds of f on $[a, b]$, we have

$$m \leq m_i \leq M_i \leq M$$

$$\Rightarrow \qquad m\Delta\alpha_i \leq m_i \Delta\alpha_i \leq M_i \Delta\alpha_i \leq M\Delta\alpha_i, \ \Delta\alpha_i \geq 0$$

Putting $i = 1, 2, ..., n$ and adding all inequalities, we get

$$m\{\alpha(b) - \alpha(a)\} \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq M\{\alpha(b) - \alpha(a)\} \qquad ...(1)$$

As in Riemann integration, § 1.1, we define two integrals, which always exist by a similar reasoning,

$$\bar{\int}_a^b f\ d\alpha = \inf . U(P, f, \alpha)$$

$$\underline{\int}_a^b f\ d\alpha = \sup . L(P, f, \alpha) \qquad ...(2)$$

the infimum and supremum being taken over all partitions of $[a, b]$. These are respectively called the **upper** and the **lower** integrals of f with respect to α.

These two integrals may or may not be equal. In cases these two integrals are equal, *i.e.*,

$$\bar{\int}_a^b f\ d\alpha = \underline{\int}_a^b f\ d\alpha,$$

we say that f is integrable with respect to α in the Riemann sense and write $f \in \mathbf{R}(\alpha)$, their common value is denoted by

$$\int_a^b f\ d\alpha$$

or sometimes by

$$\int_a^b f(x)\ d\alpha(x),$$

and is called the *Riemann-Stieltjes integral* (or simply the *Stieltjes integral*) of f with respect to α, over $[a, b]$.

From (1) and (2), it follows that

$$m\{\alpha(b) - \alpha(a)\} \le L(P, f, \alpha) \le \underline{\int}_a^b f\ d\alpha \le \bar{\int}_a^b f\ d\alpha$$

$$\le U(P, f, \alpha) \le m\{\alpha(b) - \alpha(a)\}. \qquad ...(3)$$

Remark. The upper and the lower integrals always exist for bounded functions but these may not be equal for all bounded functions. Such functions are not integrable. Thus, the question of their equality and hence that of the integrability of the function is our main concern.

The Riemann-Stieltjes integral reduces to Riemann integral when $\alpha(x) = x$.

10.1.1 Some Deductions

(*i*) If $f \in \mathbf{R}(\alpha)$, then $\exists$ a number λ lying between the bounds of f such that

$$\int_a^b f\ d\alpha = \lambda\{\alpha(b) - \alpha(a)\} \quad \text{(using 3)}$$

(*ii*) If f is continuous on $[a, b]$, then $\exists$ a number $\xi \in [a, b]$, such that

$$\int_a^b f\ d\alpha = f(\xi)\ \{\alpha(b) - \alpha(a)\}$$

(*iii*) If $f \in \mathbf{R}(\alpha)$, and k is a number, such that

$$|f(x)| \le k, \text{ for all } x \in [a, b],$$

then

$$\left| \int_a^b f \, d\alpha \right| \le k\{\alpha(b) - \alpha(a)\}$$

(*iv*) If $f \in \mathbf{R}(\alpha)$ over $[a, b]$ and $f(x) \ge 0$, for all $x \in [a, b]$, then

$$\int_a^b f \, d\alpha \begin{cases} \ge 0, & b \ge a \\ \le 0, & b \le a \end{cases}$$

Since $f(x) \ge 0$, the lower bound $m \ge 0$ and therefore, the result follows from (3).

(*v*) If $f \in \mathbf{R}(\alpha)$, $g \in \mathbf{R}(\alpha)$ over $[a, b]$, such that $f(x) \ge g(x)$, then

$$\int_a^b f \, d\alpha \ge \int_a^b f \, d\alpha, \, b \ge a,$$

and

$$\int_a^b f \, d\alpha \le \int_a^b g \, d\alpha, \, b \le a$$

The result follows by reasoning similar to that of Deduction 5 § 9.2 , Chapter 9.

10.1.2 Refinement of Partitions

Theorem 10.1. *If P^* is a refinement of P, then*

(*i*) $L(P^*, f, \alpha) \ge L(P, f, \alpha)$, *and*

(*ii*) $L(P^*, f, \alpha) \le U(P, f, \alpha)$.

Let us prove (*ii*).

Let $P = \{a = x_0, x_1, ..., x_n = b\}$ be a partition of the given interval. Suppose first that P^* contains just one point more than P. Let this extra point ξ belongs to Δx_i, *i.e.*, $x_{i-1} < \xi < x_i$.

As f is bounded over the entire interval $[a, b]$, it is bounded on every sub-interval Δx_i $(i = 1, 2, ..., n)$. Let W_1, W_2, M_i be the upper bounds (supremum) of f in the intervals $[x_{i-1}, \xi]$, $[\xi, x_i]$, $[x_{i-1}, x_i]$, respectively.

Clearly

$$W_1 \le M_i, \; W_2 \le M_i.$$

Hence $$U(P^*, f, \alpha) - U(P, f, \alpha) = W_1\{\alpha(\xi) - \alpha(x_{i-1})\} + W_2\{\alpha(x_i) - \alpha(\xi)\} - M_i\{\alpha(x_i) - \alpha(x_{i-1})\}.$$

$$= (W_1 - M_i)\{\alpha(\xi) - \alpha(x_{i-1})\} + (W_2 - M_i)\{\alpha(x_i) - \alpha(\xi)\} \le 0$$

$$\Rightarrow \qquad U(P^*, f, \alpha) \le U(P, f, \alpha)$$

If P^* contains m points more than P, we repeat the above reasoning m times and arrive at the result (*ii*).

The proof of (*i*) is similar.

Theorem 10.2. *For any two partitions* P_1, P_2,

$$L(P_1, f, \alpha) \le U(P_2, f, \alpha)$$

i.e., no upper sum can ever be less than any lower sum.

Corollary. For a bounded function f,

$$\underline{\int}_a^b f\, d\alpha \le \bar{\int}_a^b f\, d\alpha$$

The proofs are similar to that of Theorem 9.2, Chapter 9.

Exercise. If $P^* \supseteq P$, then show that

$$U(P^*, f, \alpha) - L(P^*, f, \alpha) \le U(P, f, \alpha) - L(P, f, \alpha).$$

10.2 A CONDITION OF INTEGRABILITY

Theorem 10.3. *A function f is integrable with respect to* α *on* $[a, b]$ *if and only if for every* $\varepsilon > 0$, *there exists a partition P of* $[a, b]$, *such that*

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

Necessary. Let $f \in \mathbf{R}(\alpha)$ over $[a, b]$.

$$\therefore \qquad \int_a^b f\, d\alpha = \bar{\int}_a^b f\, d\alpha = \underline{\int}_a^b f\, d\alpha\ .$$

Let $\varepsilon > 0$ be any number.

Since the upper and the lower integrals are the infimum and the supremum, respectively, of the upper and the lower sums, therefore, $\exists$ partitions P_1 and P_2 such that

$$U(P_1, f, \alpha) < \bar{\int}_a^b f\, d\alpha + \frac{1}{2}\varepsilon = \int_a^b f\, d\alpha + \frac{1}{2}\varepsilon\ ,$$

$$L(P_2, f, \alpha) > \bar{\int}_a^b f\, d\alpha - \frac{1}{2}\varepsilon = \int_a^b f\, d\alpha - \frac{1}{2}\varepsilon\ .$$

Let $P = P_1 \cup P_2$ be the common refinement of P_1 and P_2.

$$\therefore \qquad U(P, f, \alpha) \le U(P_1, f, \alpha)$$

$$< \int_a^b f\, d\alpha + \frac{1}{2}\varepsilon < L(P_2, f, \alpha) + \varepsilon \le L(P, f, \alpha) + \varepsilon$$

$$\Rightarrow \qquad U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$

Sufficient. For $\varepsilon > 0$, let P be a partition for which

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

For any partition P, we know that

$$L(P, f, \alpha) \le \underline{\int_a^b} f\, d\alpha \le \overline{\int_a^b} f\, d\alpha \le U(P, f, \alpha)$$

$$\therefore \qquad \overline{\int} f\, d\alpha - \underline{\int} f\, d\alpha \le U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

But a non-negative number can be less than every positive number, if it is zero.

$$\therefore \qquad \overline{\int_a^b} f\, d\alpha = \underline{\int_a^b} f\, d\alpha = \int_a^b f\, d\alpha$$

so that $f \in \mathbf{R}(\alpha)$, over $[a, b]$.

10.3 SOME THEOREMS

(*a*) *If* $f_1 \in \mathbf{R}(\alpha)$ *and* $f_2 \in \mathbf{R}(\alpha)$ *on* $[a, b]$, *then*

$$f_1 + f_2 \in \mathbf{R}(\alpha) \text{ and } \int_a^b (f_1 + f_2)\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha.$$

(*b*) *If* $f \in \mathbf{R}(\alpha)$, *and c is a constant, then*

$$cf \in \mathbf{R}(\alpha) \text{ and } \int_a^b cf\, d\alpha = c \int_a^b f\, d\alpha.$$

(*c*) *If* $f_1 \in \mathbf{R}(\alpha)$, $f_2 \in \mathbf{R}(\alpha)$ *and* $f_1(x) \le f_2(x)$ *on* $[a, b]$, *then*

$$\int_a^b f_1\, d\alpha \le \int_a^b f_2\, d\alpha.$$

(*d*) *If* $f \in \mathbf{R}(\alpha)$ *over* $[a, b]$ *and if* $a < c < b$, *then*

$f \in \mathbf{R}(\alpha)$ *on* $[a, c]$, *and on* $[c, b]$

and $\int_a^c f\, d\alpha + \int_c^b f\, d\alpha = \int_a^b f\, d\alpha.$

(*e*) *If* $f \in \mathbf{R}(\alpha)$ *on* $[a, b]$, *then*

$$|f| \in \mathbf{R}(\alpha) \text{ and } \left| \int_a^b f\, d\alpha \right| \le \int_a^b |f|\, d\alpha.$$

(*f*) *If* $f \in \mathbf{R}(\alpha)$ *on* $[a, b]$, *then*

$$f^2 \in \mathbf{R}(\alpha)$$

(*g*) *If* $f \in \mathbf{R}(\alpha_1)$ *and* $f \in \mathbf{R}(\alpha_2)$, *then*

$$f \in \mathbf{R}(\alpha_1 + \alpha_2) \text{ and } \int_a^b f \, d(\alpha_1 + \alpha_2) = \int_a^b f \, d\alpha_1 + \int_a^b f \, d\alpha_2$$

and if $f \in \mathbf{R}(\alpha)$ *and c a positive constant, then*

$$f \in \mathbf{R}(c\alpha) \text{ and } \int_a^b f \, d(c\alpha) = c \int_a^b f \, d\alpha.$$

(*a*) Let $f = f_1 + f_2$.

Cleary f is bounded on $[a, b]$.

If $P = \{a = x_0, x_1 ..., x_n = b\}$ be any partition of $[a, b]$ and m_i'; M_i'; m_i''; M_i''; m_i; M_i; the bounds of f_1, f_2 and f, respectively, on Δx_i, then

$$m_i' + m_i'' \le m_i \le M_i \le M_i' + M_i''$$

Multiplying by $\Delta\alpha_i$ and adding all these inequalities for $i = 1, 2, 3, ..., n$, we get

$$L(P, f_1, \alpha) + L(P, f_2, \alpha) \le L(P, f, \alpha) \le U(P, f, \alpha)$$
$$\le U(P, f_1, \alpha) + U(P, f_2, \alpha) \qquad ...(1)$$

Let $\varepsilon > 0$ be any number.

Since $f_1 \in \mathbf{R}(\alpha), f_2 \in \mathbf{R}(\alpha)$, therefore $\exists$ partitions P_1, P_2 such that

$$U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha) < \frac{1}{2}\varepsilon$$

$$U(P_2, f_2, \alpha) - L(P_2, f_2, \alpha) < \frac{1}{2}\varepsilon$$

Let $P = P_1 \cup P_2$, a refinement of P_1 and P_2.

$\therefore$

$$U(P, f_1, \alpha) - L(P, f_1, \alpha) < \frac{1}{2}\varepsilon$$

$$U(P, f_2, \alpha) - L(P, f_2, \alpha) < \frac{1}{2}\varepsilon \qquad ...(2)$$

Thus for partition P, we get from (1) and (2),

$$U(P, f, \alpha) - L(P, f, \alpha) \le U(P, f_1, \alpha) + U(P, f_2, \alpha) - L(P, f_1, \alpha) - L(P, f_2, \alpha)$$
$$\le \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

$\Rightarrow$ $f \in \mathbf{R}(\alpha)$ on $[a, b]$

Let us now proceed to prove the second part.

Since the upper integral is the infimum of the upper sums, therefore, $\exists$ partitions P_1, P_2 such that

$$U(P_1, f_1, \alpha) < \int_a^b f_1 \, d\alpha + \frac{1}{2}\varepsilon$$

$$U(P_2, f_2, \alpha) < \int_a^b f_2 \, d\alpha + \frac{1}{2}\varepsilon$$

If $P = P_1 \cup P_2$, we have

$$\left.\begin{aligned} U(P, f_1, \alpha) &< \int_a^b f_1 \, d\alpha + \frac{1}{2}\varepsilon \\ U(P, f_2, \alpha) &< \int_a^b f_2 \, d\alpha + \frac{1}{2}\varepsilon \end{aligned}\right\} \quad ...(3)$$

For such a partition P,

$$\int_a^b f \, d\alpha \le U(P, f, \alpha) \le U(P, f_1, \alpha) + U(P, f_2, \alpha) \qquad \text{[from (1)]}$$

$$\le \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, dx + \varepsilon \qquad \text{[using (3)]}$$

Since ε is arbitrary, we get

$$\int_a^b f \, d\alpha \le \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha \qquad ...(4)$$

Proceeding with $(-f_1)$ and $(-f_2)$ instead of f_1 and f_2, we get

$$\int_a^b f \, d\alpha \ge \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha \qquad ...(5)$$

(4) and (5) give

$$\int_a^b f \, d\alpha \le \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha$$

(*g*) Since $f \in \mathbf{R}(\alpha_1)$ and $f \in \mathbf{R}(\alpha_2)$, therefore for $\varepsilon > 0$, $\exists$ partitions P_1, P_2 of $[a, b]$ such that

$$U(P_1, f, \alpha_1) - L(P_1, f, \alpha_1) < \frac{1}{2}\varepsilon$$

$$U(P_2, f, \alpha_2) - L(P_2, f, \alpha_2) < \frac{1}{2}\varepsilon$$

Let $P = P_1 \cup P_2$

$\therefore$

$$U(P, f, \alpha_1) - L(P, f, \alpha_1) < \frac{1}{2}\varepsilon \qquad ...(1)$$

$$U(P, f, \alpha_2) - L(P, f, \alpha_2) < \frac{1}{2}\varepsilon$$

Let the partition P be $\{a = x_0, x_1, x_2, ..., x_n = b\}$, and m_i, M_i be bounds of f in Δx_i.

Let $\alpha = \alpha_1 + \alpha_2$.

$\therefore$

$$\alpha(x) = \alpha_1(x) + \alpha_2(x)$$

$$\Delta x_{1i} = \alpha_1(x_i) - \alpha_1(x_{i-1}),$$

$$\Delta x_{2i} = \alpha_2(x_i) - \alpha_2(x_{i-1})$$

$$\therefore \quad \Delta x_i = \alpha(x_i) - \alpha(x_{i-1})$$

$$= \alpha_1(x_i) + \alpha_2(x_i) - \alpha_1(x_{i-1}) - \alpha_2(x_{i-1})$$

$$= \Delta\alpha_{1i} + \Delta\alpha_{2i}$$

$$\therefore \quad U(P, f, \alpha) = \sum_i M_i \Delta\alpha_i$$

$$= \sum_i M_i (\Delta\alpha_{1i} + \Delta\alpha_{2i})$$

$$= U(P, f, \alpha_1) + U(P, f, \alpha_2) \qquad ...(2)$$

Similarly,

$$L(P, f, \alpha) = L(P, f, \alpha_1) + L(P, f, \alpha_2) \qquad ...(3)$$

$$\therefore \quad U(P, f, \alpha) - L(P, f, \alpha) = U(P, f, \alpha_1) - L(P, f, \alpha_1)$$

$$+ U(P, f, \alpha_2) - L(P, f, \alpha_2)$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon \text{ [using (1)]}$$

$$\Rightarrow \quad f \in \mathbf{R}(\alpha), \text{ where } \alpha = \alpha_1 + \alpha_2$$

Now to prove the second part, we notice that

$$\int_a^b f\, d\alpha = \inf U(P, f, \alpha)$$

$$= \inf \{U(P, f, \alpha_1) + U(P, f, \alpha_2)\}$$

$$\geq \inf U(P, f, \alpha_1) + \inf U(P, f, \alpha_2)$$

$$= \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2 \qquad ...(4)$$

Similarly,

$$\int_a^b f\, d\alpha = \sup L(P, f, \alpha)$$

$$\leq \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2 \qquad ...(5)$$

From (4) and (5),

$$\int_a^b f\, d\alpha = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2$$

where $\alpha = \alpha_1 + \alpha_2$.

The proofs of the remaining parts are so similar to the above proofs and virtually identical to those of the corresponding theorems for Riemann integral that it is a mere repetition and are therefore left to the reader.

Corollary. If $f_1 \in \mathbf{R}(\alpha)$ and $f_2 \in \mathbf{R}(\alpha)$ on $[a, b]$, then

$$f_1 f_2 \in \mathbf{R}(\alpha)$$

We know that if f_1, f_2 are integrable then $f_1 + f_2,\ f_1 - f_2,\ f_1^2,\ f_2^2,$ are all integrable.

Also, then $(f_1 + f_2)^2, (f_1 - f_2)^2$ are integrable.

Now

$$4 f_1 f_2 = (f_1 + f_2)^2 - (f_2 - f_2)^2$$

$$\Rightarrow f_1 f_2 \in \mathbf{R}(\alpha)$$

10.4 A DEFINITION (Integral as a Limit of Sum)

As an analog to the Riemann sum, we introduce a sum which will lead to a sufficient condition for the existence of a Riemann-Stieltjes integral.

Definition. Corresponding to a partition P of $[a, b]$ and $t_i \in \Delta x_i,$ consider the sum

$$S(P, f, \alpha) = \sum_{i=1}^{n} f(t_i)\, \Delta\alpha_i$$

We say that $S(P, f, \alpha)$ converges to A as $\mu(P) \to 0,$ *i.e.*,

$$\lim_{\mu(P) \to 0} S(P, f, \alpha) = A$$

if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|S(P, f, \alpha) - A| < \varepsilon,$ for every partition $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$, of $[a, b]$, with mesh $\mu(P) < \delta$ and every choice of t_i in Δx_i.

Theorem 10.4. *If lim* $S(P, f, \alpha)$ *exists as* $\mu(P) \to 0,$ *then*

$$f \in \mathbf{R}(\alpha), \textit{ and } \lim_{\mu(p) \to 0} S(P, f, \alpha) = \int_a^b f\, d\alpha.$$

Let us suppose that lim $S(P, f, \alpha)$ exists as $\mu(P) \to 0$ and is equal to A.

Therefore for $\varepsilon > 0, \exists\ \delta > 0$ such that for every partition P of $[a, b]$ with mesh $\mu(P) \to \delta$ and every choice of t_i and $\Delta x_i,$ we have

$$|S(P, f, \alpha) - A| < \frac{1}{2}\varepsilon$$

or
$$A - \frac{1}{2}\varepsilon < S(P, f, \alpha) < A + \frac{1}{2}\varepsilon \qquad ...(1)$$

Let P be one such partition. If we let the points t_i range over the intervals Δx_i and take the infimum and the supremum of the sums $S(P, f, \alpha),$ (1) yields

$$A - \frac{1}{2}\varepsilon \le L(P, f, \alpha) \le U(P, f, \alpha) \le A + \frac{1}{2}\varepsilon \qquad ...(2)$$

$\Rightarrow$ $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$

$\Rightarrow$ $f \in \mathbf{R}(\alpha)$ on $[a, b]$

Again, since $S(P, f, \alpha)$ and $\int_a^b f\, d\alpha$ lie between $U(P, f, \alpha)$ and $L(P, f, \alpha)$

$\therefore$ $\left| S(P, f, \alpha) - \int_a^b f\, d\alpha \right| \le U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$

$\Rightarrow$ $\lim_{\mu(p)\to 0} S(P, f, \alpha) = \int_a^b f\, d\alpha$

Remark. The theorem asserts that the existence of the limit of $S(P, f, \alpha)$ implies that $f \in \mathbf{R}(\alpha)$. The existence of the limit is a sufficient condition for $f \in \mathbf{R}(\alpha)$ but as shown in Example 10.3 it is not a necessary condition, *i.e.*, functions exist which are integrable but for which limit of $S(P, f, \alpha)$ does not exist. Thus whenever lim $S(P, f, \alpha)$ exists, it is equal to $\int f\, d\alpha$. But when $f \in \mathbf{R}(\alpha)$ nothing can be said about the existence of lim $S(P, f, \alpha)$.

Theorem 10.5. *If f is continuous on* $[a, b]$ *then* $f \in \mathbf{R}(\alpha)$ *on* $[a, b]$. *Moreover, to every* $\varepsilon > 0$ *there corresponds a* $\delta > 0$ *such that*

$$\left| S(P, f, \alpha) - \int_a^b f\, d\alpha \right| < \varepsilon$$

for every partition $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ *of* $[a, b]$ *with* $\mu(P) < \delta$, *and for every choice of* t_i *in* Δx_i, *i.e.*,

$$\lim_{\mu(p)\to 0} S(P, f, \alpha) = \int_a^b f\, d\alpha .$$

[We still assume that all functions are bounded and α is monotonic increasing.]

Let $\varepsilon > 0$ be given, and let us choose $\eta > 0$ such that

$$\eta\{\alpha(b) - \alpha(a)\} < \varepsilon \qquad ...(1)$$

Since continuity of f on the closed interval $[a, b]$ implies its uniform continuity on $[a, b]$ therefore for $\eta > 0$ there corresponds $\delta > 0$ such that

$$|f(t_1) - f(t_2)| < \eta, \text{ if } |t_1 - t_2| < \delta, t_1, t_2 \in [a, b] \qquad ...(2)$$

Let P be a partition of $[a, b]$, with norm $\mu(P) < \delta$.

Then in view of (2),

$$M_i - m_i \le \eta, \ i = 1, 2, ..., n$$

$$\therefore \quad U(P, f, \alpha) - L(P, f, \alpha) = \sum_i (M_i - m_i)\, \Delta x_i$$

$$\le \eta \sum_i \Delta x_i$$

$$= \eta\{\alpha(b) - \alpha(a)\} < \varepsilon \qquad ...(3)$$

$$\Rightarrow f \in \mathbf{R}(\alpha) \text{ over } [a, b].$$

Again if $f \in \mathbf{R}(\alpha)$, then for $\varepsilon > 0, \exists\, \delta > 0$ such that for all partitions P with $\mu(P) < \delta$,

$$\left| U(P, f, \alpha) - L(P, f, \alpha) \right| < \varepsilon$$

Since $S(P, f, \alpha)$ and $\int_a^b f\, d\alpha$ both lie between $U(P, f, \alpha)$ and $L(P, f, \alpha)$ for all partitions P with $\mu(P) < \delta$ and for all positions of t_i in Δx_i.

$$\therefore \quad \left| S(P, f, \alpha) - \int_a^b f\, d\alpha \right| < U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$

$$\Rightarrow \lim_{\mu(p)\to 0} S(P, f, \alpha) = \lim_{\mu(p)\to 0} \sum_{i=1}^{n} f\left(t_i\right) \Delta\alpha_i = \int_a^b f\, d\alpha.$$

Notes 1. Continuity is a sufficient condition for integrability of a function. It is not necessary condition. Functions exist which are integrable but not continuous.

2. For continuous function f, lim $S(P, f, \alpha)$ exists and equals $\int f\, d\alpha$.

Theorem 10.6. *If f is monotonic on* $[a, b]$, *and if* α *is continuous on* $[a, b]$, *then* $f \in \mathbf{R}(\alpha)$.

[Monotonicity of α is a still assumed.]

Let $\varepsilon > 0$ be a given positive number.

For any positive integer n, choose, a partition $P = \{x_0, x_1, ..., x_n\}$ of $[a, b]$ such that

$$\Delta\alpha_i = \frac{\alpha(b) - \alpha(a)}{n}, \ i = 1, 2, ..., n$$

This is possible because α is continuous and monotonic increasing on the closed interval $[a, b]$ and thus assumes every value between its bounds, $\alpha(a)$ and $\alpha(b)$.

Let f be monotonic increasing on $[a, b]$, so that its lower and the upper bound, m_i, M_i in Δx_i are given by

$$m_i = f(x_{i-1}), \ \ M_i = f(x_i), \ i = 1, 2, ..., n$$

$$\therefore \qquad U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i)\, \Delta x_i$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} \{f(x_i) - f(x_{i-1})\}$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \{f(b) - f(a)\}$$

$$< \varepsilon, \text{ for large } n$$

$$\Rightarrow \qquad f \in \mathbf{R}(\alpha) \text{ over } [a, b]$$

Note. $f \in \mathbf{R}(\alpha)$, *i.e.*, $\int f\, d\alpha$ exists when either

(*i*) f is continuous and α is monotonic, or

(*ii*) f is monotonic and α is continuous; of course α is still monotonic.

10.4.1 Some Examples

Example 10.1. A function α increases on $[a, b]$ and is continuous at x' where $a \le x' \le b$. Another function f is such that

$$f(x') = 1, \text{ and } f(x) = 0, \text{ for } x \ne x'$$

Prove that

$$f \in \mathbf{R}(\alpha) \text{ over } [a, b], \text{ and } \int_a^b f\, d\alpha = 0.$$

Solution. Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be a partition of $[a, b]$ and let $x' \in \Delta x_i$.

But since α is continuous at x' and increases on $[a, b]$, therefore for $\varepsilon > 0$ we can choose $\delta > 0$ such that

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) < \varepsilon, \text{ for } \Delta x_i < \delta$$

Let, P be a partition with $\mu(P) < \delta$.

Now $\quad U(P, f, \alpha) = \Delta\alpha_i$

$$L(P, f, \alpha) = 0$$

$$\therefore \qquad \bar{\int}_a^b f\, d\alpha = \inf\, U(P, f, \alpha), \text{ over all partitions } P \text{ with } \mu(P) < \delta$$

$$= 0 = \underline{\int}_a^b f d\alpha$$

$$\Rightarrow f \in \mathbf{R}(\alpha), \text{ and } \int_a^b f\, d\alpha = 0.$$

Aliter. Let $P = \{a = x_0, x_1, ..., x_n = b\}$ be a partition of $[a, b]$ and let $x' \in \Delta x_i$, $x_{i-1} \le x' < x_i$.

By continuity of α at x', for $\varepsilon > 0$, $\exists\ \delta > 0$, such that

$$|\alpha(x) - \alpha(x')| < \frac{1}{2}\varepsilon, \text{ for } |x - x'| < \delta$$

Again, since α is an increasing function, we have

$$\alpha(x) - \alpha(x') < \frac{1}{2}\varepsilon, \text{ for } 0 < x - x' < \delta$$

and

$$\alpha(x') - \alpha(x) < \frac{1}{2}\varepsilon, \text{ for } 0 < x' - x < \delta$$

Let P be a partition with $\mu(P) < \delta$.

$\therefore$
$$\begin{aligned}\Delta\alpha_i &= \alpha(x_i) - \alpha(x_{i-1}) \\ &= \alpha(x_i) - \alpha(x') + \alpha(x') - \alpha(x_{i-1}) \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon\end{aligned}$$

$$S(P, f, \alpha) = \sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i = f(t_i)\,\Delta\alpha_i$$

$$= \begin{cases} 0, & t_i \neq x' \\ \Delta\alpha_i, & t_i = x' \end{cases}$$

$\therefore$
$$\begin{aligned}|S(P, f, \alpha)| &= 0, \quad \text{when } t_i \neq x' \\ &< \varepsilon, \quad \text{when } t_i = x'\end{aligned}$$

In either case

$$\lim_{\mu(P)\to 0} S(P, f, \alpha) = 0$$

$\Rightarrow$
$$f \in \mathbf{R}(\alpha) \text{ over } [a, b], \text{ and } \int_a^b f\,d\alpha = 0.$$

Example 10.2. f is a function bounded on $[-1, 1]$, and three functions $\beta_1, \beta_2, \beta_3$ are defined as follows:

$$\beta_1(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases}$$

$$\beta_2(x) = \begin{cases} 0, & x < 0 \\ 1, & x \ge 0 \end{cases}$$

$$\beta_3(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & x = 0 \\ 1, & x > 0 \end{cases}$$

Prove that $f \in \mathbf{R}(\beta_3)$ iff f is continuous at $x = 0$, and then

$$\int_{-1}^{1} f\,d\beta_3 = f(0).$$

Solution. Let $P = \{-1 = x_0, x_1, ..., x_{i-2}, 0 = x_{i-1}, x_i, ..., x_n = 1\}$ be a partition of $[-1, 1]$ such that $x_{i-1} = 0$. Let $t_i \in \Delta x_i$.

Now

$$S(P, f, \beta_3) = \sum_{j=1}^{n} f(t_j)\{\beta_3(x_j) - \beta_3(x_{j-1})\}$$

$$= f(t_{i-1}) \cdot \frac{1}{2} + f(t_i) \cdot \left(1 - \frac{1}{2}\right)$$

$$= \frac{1}{2}\{f(t_{i-1}) + f(t_i)\} \qquad ...(1)$$

$$= f(0) \text{ in particular when } t_{i-1} = 0 = t_i \qquad ...(2)$$

Clearly t_{i-1} tends to 0 from below and t_i from above, when the norm $\mu(P)$ tends to zero.

Hence, $\lim_{\mu(P)\to 0} S(P, f, \alpha)$ exists when both the limits, $\lim_{t_{i-1}\to -0} f(t_{i-1})$ and $\lim_{t_{i\to}+0} f(t_i)$ or equivalently $\lim_{x\to 0-0} f(x)$ and $\lim_{x\to 0+0} f(x)$, exist, *i.e.*, both $f(0-)$ and $f(0+)$ exist.

Moreover, from (2) it is evident that these limits are each equal to $f(0)$. In that case

$$\lim_{\mu(P)\to 0} S(P, f, \alpha) = f(0)$$

Hence $f \in \mathbf{R}(\beta_3)$ if $f(0+) = f(0-) = f(0)$, *i.e.*, if the function f is continuous at zero and in that case

$$\int_{-1}^{1} f \, d\beta_3 = f(0)$$

Also it is clear that f is continuous if $\lim S(P, f, \alpha)$ exists. Hence $f \in \mathbf{R}(\beta_3)$ iff f is continuous at $x = 0$.

Example 10.3. For the functions β_1 and β_2 defined in Example 10.2, prove that $\beta_2 \in \mathbf{R}(\beta_1)$, although $\lim S(P, \beta_2, \beta_1)$ does not exist, as $\mu(P) \to 0$.

Solution. Let $P = \{-1 = x_0, x_1, ..., x_n = 1\}$ be a partition of $[-1, 1]$ such that $0 \in \Delta x_r$

Let $t_i \in \Delta x_i$, when $i = 1, 2, 3, ..., n$.

Now
$$S(P, \beta_2, \beta_1) = \sum_{i=1}^{n} \beta_2(t_i)\{\beta_1(x_i) - \beta_1(x_{i-1})\}$$

$$= \beta_2(t_r)$$

$\therefore \lim_{\mu(P)\to 0} S(P, \beta_2, \beta_1) = 0$ or 1, according as $t_r < 0$ or ≥ 0

Thus $\lim S(P, \beta_2, \beta_1)$ does not exist.

Let $P^* = P \cup \{0\}$, and $0 \in \Delta x_r$.

Now

$$U(P^*, \beta_2, \beta_1) = 1 \cdot \{\beta_1(x_r) - \beta_1(0)\} = 1$$
$$L(P^*, \beta_2, \beta_1) = 1 \cdot \{\beta_1(x_r) - \beta_1(0)\} = 1$$

Thus

$$U(P^*, \beta_2, \beta_1) = L(P^*, \beta_2, \beta_1) = 1$$

$\therefore$ $$\beta_2 \in \mathbf{R}(\beta_1) \text{ and } \int_{-1}^{1} \beta_2 \, d\beta_1 = 1$$

EXERCISE

1. For the functions f, β_1, β_2 defined in Example 10.2, prove that

   (*a*) $f \in \mathbf{R}(\beta_1)$ iff $f(0+) = f(0)$
   and in that case
   $$\int_{-1}^{1} f \, d\beta_1 = f(0).$$

   (*b*) $f \in \mathbf{R}(\beta_2)$ iff $f(0-) = f(0)$,
   and in that case
   $$\int_{-1}^{1} f \, d\beta_2 = f(0).$$

2. Show that
   $$\int_{0}^{4} x \, d([x] - x) = \frac{3}{2}$$
   where $[x]$ is the greatest integer not exceeding x.

10.5 SOME IMPORTANT THEOREMS

We add a few theorems before closing the discussion.

Theorem 10.7 (*A particular case*). *If f is continuous on $[a, b]$ and α has a continuous derivative on $[a, b]$, then*

$$\int_a^b f \, d\alpha = \int_a^b f \alpha' \, dx.$$

Under the given conditions all the integrals exist.

Let $P = \{a = x_0, ..., x_n = b\}$ be any partition of $[a, b]$. Thus, by Lagrange's Mean Value Theorem it is possible to find $t_i \in]x_{i-1}, x_i[$, such that

$$\alpha(x_i) - \alpha(x_{i-1}) = \alpha'(t_i)\,(x_i - x_{i-1}),\ i = 1, 2, ..., n$$

or

$$\Delta\alpha_i = \alpha'(t_i)\,\Delta x_i$$

$$\therefore \qquad S(P, f, \alpha) = \sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i$$

$$= \sum_{i=1}^{n} f(t_i)\,\alpha'(t_i)\,\Delta x_i = S(P, f\alpha') \qquad \ldots(6)$$

Proceeding to limits as $\mu(P) \to 0$, since both the limits exist, we get

$$\int_a^b f\,d\alpha = \int_a^b f\alpha'\,dx$$

Notes 1. The theorem illustrates one of the situations in which Riemann-Stieltjes integrals reduce to Riemann integrals.

2. In equation (6) $\lim S(P, f, \alpha)$ exists in view of Theorem 10.5 while $\lim S(P, f\alpha')$ exists because $f\alpha'$ is continuous and hence integrable in the Riemann sense.

Theorem 10.8. (*First Mean Value Theorem*). *If a function f is continuous on* $[a, b]$ *and* α *is monotonic increasing on* $[a, b]$, *then there exists a number* ξ *in* $[a, b]$ *such that*

$$\int_a^b f\,d\alpha = f(\xi)\,\{\alpha(b) - \alpha(a)\}.$$

f is continuous and α is monotonic, therefore $f \in \mathbf{R}(\alpha)$.

Let m, M be the infimum and supremum of f in $[a, b]$. Then as in § 1.1,

$$m\{\alpha(b) - \alpha(a)\} \le \int_a^b f\,d\alpha \le M\{\alpha(b) - \alpha(a)\}$$

Hence, there exists a number μ, $m \le \mu \le M$ such that

$$\int_a^b f\,d\alpha = \mu\{\alpha(b) - \alpha(a)\}$$

Again, since f is continuous, there exists a number $\xi \in [a, b]$ such that $f(\xi) = \mu$

$$\therefore \qquad \int_a^b f\,d\alpha = f(\xi)\,\{\alpha(b) - \alpha(a)\}.$$

Remark. It may not be possible always to choose ξ such that $a < \xi < b$.

Consider $\alpha(x) = \begin{cases} 0, & x = a \\ 1, & a < x \le b \end{cases}$

For a continuous function f, we have

$$\int_a^b f\, d\alpha = f(a) = f(a)\,\{\alpha(b) - \alpha(a)\}$$

Theorem 10.9. *If f is continuous and α monotone on $[a, b]$, then*

$$\int_a^b f\, d\alpha = \left[f(x)\alpha(x)\right]_a^b - \int_a^b \alpha\, df\,.$$

Under the given conditions all the integrals exist by Theorem 10.5.

Let $P = \{a = x_0, x_1, ..., x_n = b\}$ be a partition of $[a, b]$.

Choose $t_1, t_2, ..., t_n$ such that $x_{i-1} \le t_i \le x_i$, and let $t_0 = a$, $t_{n+1} = b$, so that $t_{i-1} \le x_{i-1} \le t_i$.

Clearly $Q = \{a, = t_0, t_1, t_2, ..., t_n, t_{n+1} = b\}$ is also a partition of $[a, b]$.

Now

$$\begin{aligned} S(P, f, \alpha) &= \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \\ &= f(t_1)[\alpha(x_1) - \alpha(x_0)] + f(t_2)[\alpha(x_2) - \alpha(x_1)] + ... \\ &\quad + f(t_n)[\alpha(x_n) - \alpha(x_{n-1})] \\ &= -\alpha(x_0) f(t_1) - \alpha(x_1)[f(t_2) - f(t_1)] + a(x_2)[f(t_3) - f(t_2)] + ... \\ &\quad + \alpha(x_{n-1})[f(t_n) - f(t_{n-1})] + \alpha(x_n)\, f(t_n) \end{aligned}$$

Adding and subtracting $\alpha(x_0) f(t_0) + \alpha(x_n) f(t_{n+1})$, we get

$$\begin{aligned} S(P, f, \alpha) &= \alpha(x_n) f(t_{n+1}) - \alpha(x_0) f(t_0) - \sum_{i=0}^{n} \alpha(x_i)\{f(t_{i+1}) - f(t_i)\} \\ &= f(b)\alpha(b) - f(a)\,\alpha(a) - S(Q, \alpha, f) \qquad ...(1) \end{aligned}$$

If $\mu(P) \to 0$, then $\mu(Q) \to 0$ and Theorem 10.5 shows that $\lim S(P, f, \alpha)$ and $\lim S(Q, \alpha, f)$ both exist and that

$$\lim S(P, f, \alpha) = \int_a^b f\, d\alpha$$

and

$$\lim S(Q, \alpha, f) = \int_a^b \alpha\, df$$

Hence proceeding to limits when $\mu(P) \to 0$, we get from (1),

$$\int_a^b f\, d\alpha = \left[f(x)\,\alpha(x)\right]_a^b - \int_a^b \alpha\, df \qquad ...(2)$$

where $\left[f(x)\,\alpha(x)\right]_a^b$ denotes the difference $f(b)\alpha(b) - f(a)\alpha(a)$.

Remark. The theorem holds when one of the functions is continuous and the other monotone.
Note. The theorem is similar to the theorem, '*Integration by parts*' for Riemann integration.

Corollary. The result of the theorem can be put in a slightly different form, by using Theorem 10.9, if, in addition to monotonicity α is continuous also

$$\begin{aligned}\int_a^b f\,d\alpha &= f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha\,df \\ &= f(b)\alpha(b) - f(a)\alpha(a) - \alpha(\xi)\,[f(b) - f(a)] \\ &= f(b)[\alpha(\xi) - \alpha(a)] + f(b)\,[\alpha(b) - \alpha(\xi)]\end{aligned}$$

where $\xi \in [a, b]$.

Stated in this form, it is called the **Second Mean Value Theorem.**

CHAPTER 11

Functions of Several Variables

So far attention has mainly been directed to functions of a single independent variable and the application of the differential calculus to such functions has been considered. In this chapter we shall be mainly concerned with the application of differential calculus to functions of more than one variable. The characteristic properties of a function of n independent variables may usually be understood by the study of a function of two or three variables and this restriction of two or three variables will be generally maintained. This restriction has the considerable advantage of simplifying the formulae and of reducing the mechanical labour.

11.1 EXPLICIT AND IMPLICIT FUNCTIONS

If we consider a set of n independent variables $x, y, z, ..., t$ and one dependent variable u, the equation

$$u = f(x, y, z, ..., t) \qquad ...(1)$$

denotes the functional relation. In this case if $x_1, y_1, z_1, ..., t_1$, are the n arbitrarily assigned values of the independent variables, the corresponding values of the dependent variable u are determined by the functional relation.

The function represented by equation (1) is an *explicit* function but where several variables are concerned it is rarely possible to obtain an equation expressing one of the variables explicitly in terms of the others. Thus, most of the functions of more than one variable are *implicit* functions, that is to say we are given a functional relation.

$$\phi(x, y, z, ..., t) = 0 \qquad ...(2)$$

connecting the n variables $x, y, z, ..., t$, and is not in general possible to solve this equation to find an *explicit* function which expresses one of these variables say x, in terms of the other $n - 1$ variables.

In this chapter, we shall be mainly concerned with the *explicit* functions.

11.1.1 An Explicit Function of Two Variables

If x, y are two independent variables and a variable z depends for its values on the values of x, y by a functional relation

$$z = f(x, y) \qquad ...(3)$$

then we say z is a *function of* x, y. The ordered pair of numbers (x, y) is called a *point* and the aggregate of the pairs of numbers (x, y) is said to be the *domain* (or region) *of definition* of the function.

When the domain of definition is bounded by a closed curve C, it is said to be *closed* if f is defined for all points within and on the curve C; but *open or unclosed* when the function is defined for points within but not on the curve C.

11.1.2 The Neighbourhood of a Point

The set of values x_1, y_1 other than a, b that satisfy the conditions

$$|x_1 - a| < \delta, |y_1 - b| < \delta$$

where δ is an arbitrarily small positive number, is said to form a *neighbourhood* of the point (a, b). Thus a neighbourhood is the square

$$(a - \delta, a + \delta; b - \delta, b + \delta)$$

where x takes any value from $a - \delta$ to $a + \delta$ except a, and y from $b - \delta$ to $b + \delta$ except b.

This is not the only way of specifying a neighbourhood of a point. There can be many other, though equivalent ways; for example the points inside the circle $x^2 + y^2 = \delta^2$ may be taken as a neighbourhood of the point (0, 0).

11.1.3 Limit Point

A point (ξ, η) is called a *limit point* or *a point of condensation* of a set of points S, if every neighbourhood of (ξ, η) contains an infinite number of points of S. The limit point itself may or may not be a point of the set. For example, the point (0, 0) is a limit point of the set $\{(1/m, 1/n) : m, n \in \mathbf{N}\}$.

11.1.4 The Limit of a Function

A function f is said to tend to a limit l as a point (x, y) tends to the point (a, b) if to every arbitrarily small positive number ε, there corresponds a positive number δ, such that

$$|f(x, y) - l| < \varepsilon,$$

for every point (x, y), [different from (a, b)] which satisfies

$$|x - a| < \delta, |y - b| < \delta$$

In other words, *a function f tends to a limit l, when* (x, y) *tends to* (a, b) *if to every positive number* ε, *there corresponds a neighbourhood N of* (a, b) *such that*

$$|f(x, y) - l| < \varepsilon,$$

for every point (x, y) *other than* (a, b) *of the neighbourhood N.*

Symbolically we then write

$$\lim_{(x, y) \to (a, b)} f(x, y) = l.$$

l is the *limit* (the *double limit* or the *simultaneous limit*) of f when x, y tend to a, b simultaneously.

Remark. The above definition implies that there must be no assumption of any relation between the independent variables as they tend to their respective limits.

For instance take $f(x, y)$ where

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

and find the limit when $(x, y) \to (0, 0)$.

If we put $y = m_1x$ and let $x \to 0$, we get the limit to be equal to $\frac{m_1}{1+m_1^2}$, while putting $y = m_2x$ leads to a limit $\frac{m_2}{1+m_2^2}$. Similarly letting $x \to 0$, while y remains constant or vice-versa leads to zero limit. Thus, we are led to erroneous results. Geometrically speaking when we approach the point (0, 0) along different paths, first along lines with slopes m_1 and m_2 and then along lines parallel to the coordinate axes, the function reaches different limits. The simultaneous limit postulates that by whatever path the point is approached, the function f attains the same limit. In general the determination whether a simultaneous limit exists or not is a difficult matter but very often a simple consideration enables us to show that the *limit does not exist.*

It may however be noted that

$$\lim_{(x,\,y)\to(a,\,b)} f(x,\,y) = l \Rightarrow \lim_{x\to a} f(x, b) = l = \lim_{y\to b} f(a,\,y).$$

Non-existence of limit. The above remark makes it amply clear that if $\lim\limits_{(x,\,y)\to(a,\,b)} f(x,\,y) = l$ and if $y = \phi(x)$ is any function such that $\phi(x) \to b$, when $x \to a$, then $\lim\limits_{x\to a} f(x, \phi(x))$, must exist and be equal to l.

Thus, if we can find two functions $\phi_1(x), \phi_2(x)$ such that the limits of $f(x, \phi_1(x))$ and $f(x, \phi_2(x))$ are different, then the simultaneous limit in question does not exist.

Example 11.1(*a*). Let

$$f(x,\,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2}, & \text{if} \quad x^4+y^2 \neq 0 \\ 0, & \text{if} \quad x+y=0 \end{cases}$$

If we approach the origin along any axis, $f(x,\,y) \equiv 0$.

If we approach (0, 0) along any line $y = mx$, then

$$f(x,\,y) = f(x, mx) = \frac{mx^3}{x^4+m^2x^2} = \frac{mx}{x^2+m^2} \to 0, \quad \text{as } x \to 0$$

So any straight line approach gives,

$$\lim_{(x,\,y)\to(0,\,0)} f(x,\,y) = 0.$$

But putting $y = mx^2$,

$$\lim_{(x,\,y)\to(0,\,0)} f(x,\,y) = \lim_{x\to 0} f(x, mx^2) = \frac{m}{1+m^2},$$

which is different for the different m selected.

Hence $\lim\limits_{(x,\,y)\to(0,\,0)} f(x,\,y)$ does not exist.

Thus the function possesses no limit at the origin, but a straight line approach gives the limit zero.

Example 11.1(*b*). Show that

$$\lim_{(x,\,y)\to(0,\,0)} \frac{2xy^2}{x^2+y^4} \text{ does not exist.}$$

Solution. If we put $x = my^2$ and let $y \to 0$, we get

$$\lim_{y\to 0} \frac{2my^4}{(m^2+1)y^4} = \frac{2m}{1+m^2}$$

which is different for different values of m.

Hence, the limit does not exist.

Remark. It is pointed out earlier also that the determination of a simultaneous limit is a difficult matter but a simple consideration, as shown above, very often, enables us to show that the limit does not exist. We now show that sometimes it is possible to determine the simultaneous limit by changing to polars.

Example 11.2(*a*). Show that

$$\lim_{(x,\,y)\to(0,\,0)} xy\frac{x^2-y^2}{x^2+y^2} = 0.$$

Solution. Put $x = r\cos\theta$, $y = r\sin\theta$,

$$\left| xy\frac{x^2-y^2}{x^2+y^2} \right| = \left| r^2 \sin\theta\cos\theta\cos 2\theta \right|$$

$$= \left| \frac{r^2}{4}\sin 4\theta \right| \le \frac{r^2}{4} = \frac{x^2+y^2}{4} < \varepsilon,$$

if

$$\frac{x^2}{4} < \frac{\varepsilon}{2}, \quad \frac{y^2}{4} < \frac{\varepsilon}{2}$$

or if

$$|x| < \sqrt{2\varepsilon} = \delta, \quad |y| < \sqrt{2\varepsilon} = \delta$$

Thus for $\varepsilon > 0, \exists\, \delta > 0$ such that

$$\left| xy\frac{x^2-y^2}{x^2+y^2} - 0 \right| < \varepsilon, \text{ when } |x| < \delta, |y| < \delta$$

$$\Rightarrow \qquad \lim_{(x,\,y)\to(0,\,0)} xy\frac{x^2-y^2}{x^2+y^2} = 0$$

Example 11.2(*b*). Show that

$$\lim_{(x,\,y)\to(0,\,0)} \frac{\sqrt{x^2y^2+1}-1}{x^2+y^2} = 0.$$

Solution. Since x, y are small

$$\frac{\sqrt{x^2y^2+1}-1}{x^2+y^2} = \frac{(1+x^2y^2)^{1/2}-1}{x^2+y^2} \simeq \frac{\frac{1}{2}x^2y^2}{x^2+y^2}$$

Now changing to polars, we can show, as in the above example, that

$$\lim_{(x,\,y)\to(0,\,0)} \frac{\frac{1}{2}x^2y^2}{x^2+y^2} = 0$$

Hence the required result.

EXERCISE

1. Show that

   (*i*) $\lim\limits_{(x,\,y)\to(0,\,0)} \left(\frac{1}{|x|}+\frac{1}{|y|}\right) = \infty$, (*ii*) $\lim\limits_{(x,\,y)\to(0,\,0)} \frac{xy^2}{x^2+y^2} = 0$,

   (*iii*) $\lim\limits_{(x,\,y)\to(0,\,0)} (x+y) = 0$, (*iv*) $\lim\limits_{(x,\,y)\to(0,\,0)} (1/xy)\sin(x^2y+xy^2) = 0$.

2. Show that the limit, when $(x,\ y) \to (0,\ 0)$ does not exist in each case

   (*i*) $\lim \frac{2xy}{x^2+y^2}$, (*ii*) $\lim \frac{xy^3}{x^2+y^6}$,

   (*iii*) $\lim \frac{x^2y^2}{x^2y^2+(x^2-y^2)^2}$, (*iv*) $\lim \frac{x^3+y^3}{x-y}$

   [*Hint*. (*iv*) Put $y = x - mx^3$].

3. Show that the limit, when $(x,\ y) \to (0,\ 0)$ exist in each case

   (*i*) $\lim \frac{xy}{\sqrt{x^2+y^2}}$, (*ii*) $\lim \frac{x^3y^3}{x^2+y^2}$,

   (*iii*) $\lim \frac{x^3-y^3}{x^2+y^2}$, (*iv*) $\lim \frac{x^4+y^4}{x^2+y^2}$.

11.1.5 Algebra of Limits

If f and g be two functions with a domain N, we define four functions, $f \pm g$, fg, f/g on N by setting

$$(f+g)(x,\,y) = f(x,\,y) + g(x,\,y)$$

$$(f - g)(x, y) = f(x, y) - g(x, y)$$

$$f \cdot g\,(x, y) = f(x, y) \cdot g(x, y)$$

$$(f/g)(x, y) = f(x, y)/g(x, y), \text{ if } g(x, y) \neq 0, \text{ for } (x, y) \in N.$$

Theorem 11.1. *If f, g be two functions defined on some neighbourhood of a point* (a, b) *such that* $\lim f(x, y) = l$, $\lim g(x, y) = m$, *when* $(x, y) \to (a, b)$, *then*

(*i*) $\lim(f + g) = \lim f + \lim g = l + m$

(*ii*) $\lim(f - g) = \lim f - \lim g = l - m$

(*iii*) $\lim(f \cdot g) = \lim f \cdot \lim g = l \cdot m$

(*iv*) $\lim \dfrac{f}{g} = \dfrac{\lim f}{\lim g} = \dfrac{l}{m}$, *provided* $m \neq 0$, *when* $(x, y) \to (a, b)$.

The proofs are exactly similar to those of the corresponding theorems for a single variable.

Example 11.3.(*a*) Prove that

$$\lim_{(x, y)\to(1, 2)} (x^2 + 2y) = 5.$$

Solution. *Method* 1. (Using definition of limit). We have to show that for any $\varepsilon > 0$, we can find $\delta > 0$, such that

$$\left| x^2 + 2y - 5 \right| < \varepsilon, \text{ when} \left| x - 1 \right| < \delta, \left| y - 2 \right| < \delta$$

If $\left| x - 1 \right| < \delta$, and $\left| y - 2 \right| < \delta$, then

$$1 - \delta < x < 1 + \delta \text{ and } 2 - \delta < y < 2 + \delta, \text{ excluding } x = 1, y = 2$$

Thus

$$1 - 2\delta + \delta^2 < x^2 < 1 + 2\delta + \delta^2$$

and

$$4 - 2\delta < 2y < 4 + 2\delta$$

Adding

$$5 - 4\delta + \delta^2 < x^2 + 2y < 5 + 4\delta + \delta^2$$

or

$$-4\delta + \delta^2 < x^2 + 2y - 5 < 4\delta + \delta^2$$

Now if $\delta \leq 1$, it follows that

$$-5\delta < x^2 + 2y - 5 < 5\delta$$

i.e.,

$$\left| x^2 + 2y - 5 \right| < 5\delta = \varepsilon$$

so that $\delta = \varepsilon/5$ (or $\delta = 1$ whichever is smaller).

$$\therefore \qquad \left| x^2 + 2y - 5 \right| < \varepsilon, \text{ when } \left| x - 1 \right| < \delta, \left| y - 2 \right| < \delta$$

$$\therefore \qquad \lim_{(x,\, y)\to(1,\, 2)} (x^2 + 2y) = 5$$

Method 2. Using above theorem on algebra of limits,

$$\lim_{(x,\, y)\to(1,\, 2)} (x^2 + 2y) = \lim_{(x,\, y)\to(1,\, 2)} x^2 + \lim_{(x,\, y)\to(1,\, 2)} 2y = 1 + 4 = 5.$$

Example 11.3.(*b*) Show that

$$(i) \quad \lim_{(x,\, y)\to(0,\, 0)} \frac{x \sin (x^2 + y^2)}{x^2 + y^2} = 0, \quad (ii) \quad \lim_{(x,\, y)\to(2,\, 1)} \frac{\sin^{-1}(xy - 2)}{\tan^{-1}(3xy - 6)} = \frac{1}{3}.$$

Solution.

$$(i) \quad \lim_{(x,\, y)\to(0,\, 0)} \frac{x \sin (x^2 + y^2)}{x^2 + y^2} = \lim_{(x,\, y)\to(0,\, 0)} x \cdot \lim_{(x,\, y)\to(0,\, 0)} \frac{\sin (x^2 + y^2)}{x^2 + y^2} = 0.1 = 0$$

$$(ii) \quad \lim_{(x,\, y)\to(2,\, 1)} \frac{\sin^{-1}(xy - 2)}{\tan^{-1}(3xy - 6)} = \lim_{t\to 0} \frac{\sin^{-1} t}{\tan^{-1} 3t}, \text{ where } t = xy - 2 = \lim_{t\to 0} \frac{1/\sqrt{1 - t^2}}{3/(1 + 9t^2)} = \frac{1}{3}$$

EXERCISE

1. Show that $\lim\limits_{(x,\, y)\to(0,\, 1)} \tan^{-1}(y/x)$, does not exist.

   $\left[\textit{Hint} : \text{Limit from the left is } -\frac{\pi}{2} \text{ and that from the right } \frac{\pi}{2} \right]$.

2. Show, by using the definition that

   $$\lim_{(x,\, y)\to(1,\, 2)} 3xy = 6$$

3. Prove that

   $$(i) \quad \lim_{(x,\, y)\to(4,\, \pi)} x^2 \sin \frac{y}{8} = 8\sqrt{2}, \qquad (ii) \quad \lim_{(x,\, y)\to(0,\, 1)} e^{-1/x^2 (y - 1)^2} = 0,$$

   $$(iii) \quad \lim_{(x,\, y)\to(0,\, 1)} \frac{x + y - 1}{\sqrt{x} - \sqrt{1 - y}} = 0, \; x > 0, \; y < 1.$$

11.1.6 Repeated Limits

If a function f is defined in some neighbourhood of (a, b), then the limit

$$\lim_{y\to b} f(x, y),$$

if it exists, is a function of x, say $\phi(x)$. If then the limit $\lim\limits_{x\to a} \phi(x)$ exists and is equal to λ, we write

$$\lim_{x\to a}\lim_{y\to b} f(x, y) = \lambda$$

and say that λ is a *repeated limit* of f as $y \to b, x \to a$.

If we change the order of taking the limits, we get the other repeated limit

$$\lim_{y\to b}\lim_{x\to a} f(x, y) = \lambda' \text{ (say)}$$

when first $x \to a$, and then $y \to b$.

These two limits may or may not be equal.

Note. In case the simultaneous limit exists, these two repeated limits if they exist are necessarily equal but the converse is not true. However *if the repeated limits are not equal, the simultaneous limit cannot exist*.

Example 11.4.(*a*) Let

$$f(x, y) = \frac{xy}{x^2 + y^2}, \text{ then}$$

$$\lim_{y\to 0}\lim_{x\to 0} f(x, y) = \lim_{y\to 0}(0) = 0,$$

$$\lim_{x\to 0}\lim_{y\to 0} f(x, y) = 0.$$

Thus the repeated limits exist and are equal. But the simultaneous limit does not exist which may be seen by putting $y = mx$.

Example 11.4.(*b*) Let

$$f(x, y) = \frac{y - x}{y + x} \cdot \frac{1 + x}{1 + y}, \text{ then}$$

$$\lim_{x\to 0}\lim_{y\to 0} f(x, y) = \lim_{x\to 0}\left(-\frac{1 + x}{1}\right) = -1,$$

$$\lim_{y\to 0}\lim_{x\to 0} f(x, y) = \lim_{y\to 0}\left(\frac{1}{1 + y}\right) = 1.$$

Thus the two repeated limits exist but are unequal, consequently the simultaneous limit cannot exist, which may be verified by putting $y = mx$.

Example 11.5. Show that the limit exists at the origin but the repeated limits do not, where

$$f(x, y) = \begin{cases} x \sin\left(\frac{1}{y}\right) + y \sin\left(\frac{1}{x}\right), & xy \neq 0 \\ 0, & xy = 0 \end{cases}$$

Solution. Here $\lim_{y\to 0} f(x, y)$, $\lim_{x\to 0} f(x, y)$ do not exist and therefore $\lim_{x\to 0}\lim_{y\to 0} f(x, y)$; $\lim_{y\to 0}\lim_{x\to 0} f(x, y)$ do not exist.

Again

$$\left| x \sin \frac{1}{y} + y \sin \frac{1}{x} \right| < |x| + |y| \leq 2(x^2 + y^2)^{1/2} < \varepsilon,$$

if

$$x^2 < \frac{\varepsilon^2}{4},\ y^2 < \frac{\varepsilon^2}{4}$$

or

$$|x| < \frac{\varepsilon}{2} = \delta,\ |y| < \frac{\varepsilon}{2} = \delta$$

Thus for $\varepsilon > 0, \exists\ \delta > 0$ such that

$$\left| x \sin \frac{1}{y} + y \sin \frac{1}{x} \right| < \varepsilon, \text{ when } |x| < \delta,\ |y| < \delta$$

$$\Rightarrow \qquad \lim_{(x,\ y) \to (0,\ 0)} \left(x \sin \frac{1}{y} + y \sin \frac{1}{x} \right) = 0.$$

Example 11.6. Show that the repeated limits exist at the origin and are equal but the simultaneous limit does not exist, where

$$f(x,\ y) = \begin{cases} 1, \text{ if } xy \neq 0 \\ 0, \text{ if } xy = 0 \end{cases}$$

Solution. Here

$$\lim_{y \to 0} f(x,\ y) = \begin{cases} 1, \text{ if } x \neq 0 \\ 0, \text{ if } x = 0 \end{cases}$$

$$\therefore \qquad \lim_{x \to 0} \lim_{y \to 0} f(x,\ y) = 1$$

Similarly,

$$\lim_{y \to 0} \lim_{x \to 0} f(x,\ y) = 1$$

Hence, the repeated limits exist and are equal.

Again, since there are points arbitrarily near (0, 0) at which f is equal to 0 and points arbitrarily near (0, 0) at which f is equal to 1, therefore, there is an $\varepsilon > 0$, such that

$$|f(x,\ y) - f(0,\ 0)| = |f(x,\ y)| \not< \varepsilon,$$

for all points in any neighbourhood of (0, 0).

Hence, $\lim\limits_{(x,\ y) \to (0,\ 0)} f(x,\ y)$ does not exist.

EXERCISE

1. Show that $\lim\limits_{(x,\,y)\to(0,\,0)} f(x, y)$ and $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x, y)$ exist, but $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x, y)$ does not, where

$$f(x, y) = \begin{cases} y + x \sin\left(\dfrac{1}{y}\right), & \text{if } y \neq 0 \\ 0, & \text{if } y = 0. \end{cases}$$

2. Show that $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x, y)$ exists, but the other repeated limit and the double limit do not exist at the origin, when

$$f(x, y) = \begin{cases} y \sin (1/x) + xy/(x^2 + y^2), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

3. Show that the repeated limits exist but the double limit does not when $(x, y) \to (0, 0)$:

(*i*) $f(x, y) = \dfrac{x - y}{x + y}$, (*ii*) $f(x, y) = \dfrac{x^2y^2}{x^4 + y^4 - x^2y^2}$

(*iii*) $f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x - y}, & x \neq y \\ 0, & x = y \end{cases}$ (*iv*) $f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2}, & x \neq y \\ 0, & x = y \end{cases}$

4. Show that the limit and the repeated limits exist when $(x, y) \to (0, 0)$:

$$f(x, y) = \begin{cases} xy\,\dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & \text{at } = (0, 0). \end{cases}$$

11.2 CONTINUITY

A function f is said to be *continuous* at a point (a, b) of its domain of definition if

$$\lim_{(x,\,y)\to(a,\,b)} f(x, y) = f(a, b).$$

In other words, a function f is said to be *continuous* at a point (a, b) of its domain of definition if for $\varepsilon > 0$, there exists a neighbourhood N of (a, b) such that

$$\left| f(x, y) - f(a, b) \right| < \varepsilon, \text{ for all } (x, y) \in N.$$

Note. The definition of continuity of a function f at a point (a, b) requires that besides (a, b), f is defined in a certain neighbourhood of (a, b) and moreover the limit of f when $(x, y) \to (a, b)$ exists and equals the value $f(a, b)$. A function which is not continuous at a point is said to be *discontinuous* thereat.

Remark. A point to be particularly noticed is that if a function of more than one variable is continuous at a point, it is continuous at that point when considered as a function of a single variable. To be more specific if a function f of two variables x, y is continuous at (a, b) then $f(x, b)$ is a continuous function of x at $x = a$ and $f(a, y)$ that of y at $y = b$.

The converse however is not true, *i.e.*, a function may be a continuous function of one variable when the others remain constant and yet not be a continuous function of all the variables.

For instance, consider a function f, where

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y}, & (x, y) \neq (0, 0) \\ 0, & \text{at } (0, 0) \end{cases}$$

The function is not continuous at (0, 0) for $\lim\limits_{(x, y)\to(0, 0)} f(x, y)$ does not exist. But

$$\lim_{x\to 0} f(x, 0) = 0 = f(0, 0), \text{ and } \lim_{y\to 0} f(0, y) = 0 = f(0, 0)$$

so that f is continuous at (0, 0), when considered as a function of a single variable x or that of y.

A function is said to be continuous in a region if it is continuous at every point of the same.

As in limits, it can be easily proved that the sum, difference, product and quotient (provided the denominator does not vanish) of two continuous functions are also continuous.

The theorems on continuity for functions of a single variable can be easily extended to functions of several variables; the proofs for some of them, except for verbal changes, are the same while for others the method is not quite the same. However, within the scope of the present work, it is not possible to discuss all of them here.

Example 11.7. Investigate the continuity at (0, 0) of

$$f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Solution. Since $\lim\limits_{(x, y)\to(0, 0)} f(x, y)$ does not exist, therefore the function is not continuous at (0, 0).

Example 11.8. Investigate for continuity at (1, 2).

$$f(x, y) = \begin{cases} x^2 + 2y, & (x, y) \neq (1, 2) \\ 0, & (x, y) = (1, 2) \end{cases}$$

Solution. Here

$$\lim_{(x, y)\to(1, 2)} f(x, y) = 5 \neq f(1, 2).$$

Hence, the function is not continuous at (1, 2).

The point (1, 2) is a *point of discontinuity* of the function.

However, if the function has the value 5 at (1, 2), it was then continuous at the point.

Remark. If, as in the above example, it is possible to so redefine the value of the function at a point of discontinuity that the new function is continuous, we say that the point is a *removable discontinuity* of the original function.

Example 11.9. Show that the function

$$f(x,\ y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & (x,\ y) \neq (0,\ 0) \\ 0, & (x,\ y) = (0,\ 0) \end{cases}$$

is continuous at the origin.

Solution. Let $x = r\cos\theta$, $y = r\sin\theta$.

$$\therefore \qquad \left| \frac{xy}{\sqrt{x^2 + y^2}} \right| = r\,|\cos\theta\ \sin\theta| \leq r = \sqrt{x^2 + y^2} < \varepsilon,$$

if

$$x^2 < \frac{\varepsilon^2}{2},\quad y^2 < \frac{\varepsilon^2}{2}$$

or, if

$$|x| < \frac{\varepsilon}{\sqrt{2}},\quad |y| < \frac{\varepsilon}{\sqrt{2}}$$

Thus

$$\left| \frac{xy}{\sqrt{x^2 + y^2}} - 0 \right| < \varepsilon, \quad \text{when } |x| < \frac{\varepsilon}{\sqrt{2}}, |y| < \frac{\varepsilon}{\sqrt{2}}$$

$$\Rightarrow \qquad \lim_{(x,\ y)\to(0,\ 0)} \frac{xy}{\sqrt{x^2 + y^2}} = 0$$

$$\therefore \qquad \lim_{(x,\ y)\to(0,\ 0)} f(x,\ y) = f(0,\ 0)$$

Hence, f is continuous at (0, 0).

EXERCISE

1. Show that the following functions are discontinuous at the origin:

(*i*) $f(x,\ y) = \begin{cases} \dfrac{1}{x^2 + y^2}, & (x,\ y) \neq (0,\ 0) \\ 0, & (x,\ y) = (0,\ 0) \end{cases}$

(*ii*) $f(x,\ y) = \dfrac{x^4 - y^4}{x^4 + y^4},\ (x,\ y) \neq (0,\ 0),\ f(0,\ 0) = 0$

(*iii*) $f(x,\ y) = \dfrac{(x^2 y^2)}{(x^4 + y^4)},\ (x,\ y) \neq (0,\ 0),\ f(0,\ 0) = 0$

2. Show that the following functions are continuous at the origin:

$$(i)\quad f(x, y) = \frac{(x^2 y^2)}{(x^2 + y^2)},\ (x, y) \neq (0, 0),\ f(0, 0) = 0.$$

$$(ii)\quad f(x, y) = \begin{cases} \dfrac{x^3 y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

3. Show that the following functions are discontinuous at (0, 0):

$$(i)\quad f(x, y) = \begin{cases} \dfrac{x^2 y}{x^3 + y^3}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

$$(ii)\quad f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x - y}, & x \neq y \\ 0, & x = y \end{cases}$$

$$(iii)\quad f(x, y) = \frac{xy^3}{x^2 + y^6},\ (x, y) \neq (0, 0),\ f(0, 0) = 0.$$

4. Discuss the following functions for continuity at (0, 0):

$$(i)\quad f(x, y) = \begin{cases} \dfrac{x^2 y}{x^3 + y^3}, & x^2 + y^2 \neq 0 \\ 0, & x + y = 0 \end{cases}$$

$$(ii)\quad f(x, y) = \begin{cases} 2xy\dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

$$(iii)\quad f(x, y) = \begin{cases} 0, & (x, y) = (2y, y) \\ \exp\{|x - 2y|/(x^2 - 4xy + 4y^2)\}, & (x, y) \neq (2y, y). \end{cases}$$

5. Show that f has a removable discontinuity at (2, 3), where

$$f(x, y) = \begin{cases} 3xy, & (x, y) \neq (2, 3) \\ 6, & (x, y) = (2, 3) \end{cases}$$

Suitably redefine the function to make it continuous.

6. Show that the function f is continuous at the origin, where

$$f(x, y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

7. Can the given functions be appropriately defined at (0, 0) in order to be continuous there?

(*i*) $f(x, y) = |x|^y$, (*ii*) $f(x, y) = \sin \frac{x}{y}$,

(*iii*) $f(x, y) = \frac{x^3 + y^3}{x^2 + y^2}$, (*iv*) $f(x, y) = x^2 \log (x^2 + y^2)$.

11.3 PARTIAL DERIVATIVES

The ordinary derivative of a function of several variables with respect to one of the independent variables, keeping all other independent variables constant is called the *partial derivative* of the function with respect to the variable. Partial derivative of $f(x, y)$ with respect to x is generally denoted by $\partial f / \partial x$ or f_x or $f_x(x, y)$, while those with respect to y are denoted by $\partial f / \partial y$ or f_y or $f_y(x, y)$.

Thus
$$\frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}$$

and

$$\frac{\partial f}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}$$

when these limits exist.

The partial derivatives at a particular point (a, b) are often denoted by

$$\left[\frac{\partial f}{\partial x}\right]_{(a, b)}, \frac{\partial f(a, b)}{\partial x} \text{ or } f_x(a, b)$$

and

$$\left[\frac{\partial f}{\partial y}\right]_{(a, b)}, \frac{\partial f(a, b)}{\partial y} \text{ or } f_y(a, b)$$

$\therefore$
$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$$
$$f_y(a, b) = \lim_{k \to 0} \frac{f(a, b + k) - f(a, b)}{k}$$

in case the limit exist.

Example 11.10. If $f(x, y) = 2x^2 - xy + 2y^2$, then find $\partial f / \partial x$ and $\partial f / \partial y$ at the point (1, 2).

Solution. Now

$$\frac{\partial f}{\partial x} = 4x - y = 2, \text{ at } (1, 2)$$
$$\frac{\partial f}{\partial y} = -x + 4y = 7, \text{ at } (1, 2)$$

Note. $f_x(1, 2)$ and $f_y(1, 2)$ have been respectively obtained from $f_x(x, y)$ and $f_y(x, y)$ be replacing (x, y) by $(1, 2)$. The procedure, though simple, is not always possible. The reader has to be on his guard.

Example 11.11. If

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

show that both the partial derivatives exist at (0, 0) but the function is not continuous thereat.

Solution. Putting $y = mx$, we see that

$$\lim_{x \to 0} f(x, y) = \frac{m}{1 + m^2}$$

so that the limit depends on the value of m, *i.e.*, on the path of approach and is different for the different paths followed and therefore does not exist. Hence, the function $f(x, y)$ is not continuous at (0, 0). Again

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(0 + h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

$$f_y(0, 0) = \lim_{k \to 0} \frac{f(0, 0 + k) - f(0, 0)}{k} = \lim_{k \to 0} \frac{0}{k} = 0$$

Notes 1. Unlike the situation for functions of one variable, the existence of the first partial derivatives at a point does not imply continuity at the point. The explanation lies in the fact that the information given by the existence of the two first partial derivatives at a point is limited. The values of f_x and f_y at a point (a, b) depend only on the values of f along two lines through (a, b) respectively parallel to the coordinate axes. This information is incomplete and tells us nothing at all about the behaviour of the function f as the point (a, b) is approached along lines not parallel to the axes. On the other hand, the continuity of f at (a, b) requires the function to tend to its value $f(a, b)$ by whatever path the point (a, b) is approached. Therefore, there is nothing surprising in the fact that the *partial derivatives may exist at a point at which the function is not even continuous*.

2. In the above example, if $x \neq y$,

$$f_x = \frac{y^3 - x^2 y}{(x^2 + y^2)^2},$$

$$f_y = \frac{x^3 - xy^2}{(x^2 + y^2)^2},$$

and $f_x(0, 0)$, $f_y(0, 0)$ cannot be computed from them by letting $x = 0$, $y = 0$.

EXERCISE

1. If $f(x, y) = x^3 y + e^{xy^2}$, find f_x and f_y.

2. If $f(x, y) = xy\dfrac{(x^2 - y^2)}{(x^2 + y^2)}$, when $x^2 + y^2 \neq 0$, and $f(0, 0) = 0$,

   show that $f_x(x, 0) = 0 = f_y(0, y)$
   $f_x(0, y) = -y, f_y(x, 0) = x.$

3. If $f(x, y) = \begin{cases} \dfrac{x^2 - xy}{x + y}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$, find $f_x(0, 0), f_y(0, 0)$.

4. If $f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x - y}, & x \neq y \\ 0, & x = y \end{cases}$, show that the function is discontinuous at the origin but possesses partial derivatives f_x and f_y at every point, including the origin.

5. If $f(x, y) = \begin{cases} xy \tan(y/x), & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$,

   show that $xf_x + yf_y = 2f$.

6. Calculate f_x, f_y, $f_x(0, 0)$, $f_y(0, 0)$ for the following:

   (*i*) $f(x, y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & x \neq 0, y \neq 0 \\ 0, & x = 0 = y \end{cases}$

   (*ii*) $f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & \text{if } x^2 + y^2 \neq 0 \\ 0, & \text{if } x = y = 0. \end{cases}$

7. Show that the function

   $$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & x^2 + y^2 \neq 0 \\ 0, & x = 0 = y \end{cases}$$

   possesses first partial derivatives everywhere, including the origin, but the function is discontinuous at the origin.

8. If $f(x, y) = \sqrt{|xy|}$, find $f_x(0, 0)$, $f_y(0, 0)$.

11.3.1 A Mean Value Theorem

If f_x exists throughout a neighbourhood of a point (a, b) and $f_y(a, b)$ exists then for any point $(a + h, b + k)$ of this neighbourhood,

$$f(a + h, b + k) - f(a, b) - hf_x(a + \theta h, b + k) + k[f_y(a, b) + \eta]$$

where $0 < \theta < 1$, and η is a function of k, tending to zero with k.

Now

$$f(a+h, b+k) - f(a, b) - f(a+h, b+k) - f(a, b+k) + f(a, b+k) - f(a, b) \quad ...(1)$$

Since f_x exists in a neighbourhood of (a, b), therefore by Lagrange's mean value theorem,

$$f(a+h, b+k) - f(a, b+k) - hf_x(a+\theta h, b+k),\ 0 < \theta < 1 \quad ...(2)$$

Also $f_y(a, b)$ exist, so that

$$\lim_{k \to 0} \frac{f(a, b+k) - f(a, b)}{k} = f_y(a, b)$$

$$\Rightarrow \qquad f(a, b+k) - f(a, b) = k[f_y(a, b) + \eta] \quad ...(3)$$

where η is a function of k and tends to zero as $k \to 0$.

From (1), (2) and (3) we get the required result.

11.3.2 A Sufficient Condition for Continuity

A sufficient condition that a function f be continuous at (a, b) *is that one of the partial derivatives exists and is bounded in a neighbourhood of* (a, b) *and that the other exists at* (a, b).

Let f_x exist and be bounded in a neighbourhood of (a, b) and let $f_y(a, b)$ exist; then for any point $(a+h, b+k)$ of this neighbourhood we have (§ 3.1)

$$f(a+h, b+k) - f(a, b) - hf_x(a+\theta h, b+k) + k[f_y(a, b) + \eta]$$

where $0 < \theta < 1$, and $\eta \to 0$ as $k \to 0$.

Proceeding to limits as $(h, k) \to (0, 0)$, since $f_x(a+\theta h, b+k)$ is bounded, we have

$$\lim_{(h, k) \to (0, 0)} f(a+h, b+k) = f(a, b)$$

$$\Rightarrow \qquad f \text{ is continuous at } (a, b).$$

Note. A sufficient condition that a function be continuous in a *closed region* is that both the partial derivatives exist and are bounded throughout the region.

11.4 DIFFERENTIABILITY

Let $(x, y), (x+\delta x, y+\delta y)$ be two neighbouring points in the domain of definition of a function f. The change δf in the function as the point changes from (x, y) to $(x+\delta x, y+\delta y)$ is given by

$$\delta f = f(x+\delta x, y+\delta y) - f(x, y)$$

The function f is said to be *differentiable* at (x, y) if the change δf can be expressed in the form

$$\delta f = A\,\delta x + B\,\delta y + \delta x\,\phi(\delta x, \delta y) + \delta y\,\psi(\delta x, \delta y) \quad ...(1)$$

where A and B are constants independent of $\delta x, \delta y$ and ϕ, ψ are functions of $\delta x, \delta y$ tending to zero as $\delta x, \delta y$ tend to zero simultaneously.

Also, $A\delta x + B\delta y$ is then called the *differential* of f at (x, y) and is denoted by df. Thus

$$df = A\delta x + B\delta y$$

From (1) when $(\delta x, \delta y) \to (0, 0)$, we get

$$f(x + \delta x,\ y + \delta y) - f(x,\ y) \to 0$$

or

$$f(x + \delta x,\ y + \delta y) \to f(x,\ y)$$

$\Rightarrow$ The function f is continuous at (x, y)

Thus every *differentiable function is continuous*

Again, from (1), when $\delta y = 0$ (*i.e.*, y remains constant)

$$\delta f = A\,\delta x + \delta x\,\phi(\delta x, 0)$$

Dividing by δx and proceeding to limits as $\delta x \to 0$, we get

$$\frac{\partial f}{\partial x} = A$$

Similarly

$$\frac{\partial f}{\partial y} = B$$

Thus the constants A and B are respectively the partial derivatives of f with respect to x and y.

Hence *a function which is differentiable at a point possesses the first order partial derivatives thereat*.

Converse, of course is not true, so that functions exist which are continuous and may even possess partial derivatives at a point but are not differentiable thereat (see example 11.12 below),

Again the differential of f is given by

$$df = A\delta x + B\delta y = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$$

Taking $f = x$, we get $dx = \delta x$.

Similarly taking $f = y$, we obtain $dy = \delta y$.

Thus the differentials dx, dy of x, y are respectively δx and δy, and

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = f_x\,dx + f_y\,dy \qquad ...(2)$$

is the differential of f at (x, y).

Notes 1. If we replace $\delta x, \delta y$, by h, k in equation (1) we say that the function is differentiable at a point (a, b) of the domain of definition if df can be expressed as

$$df = f(a + h, b + k) - f(a, b)$$
$$= Ah + Bk + h\phi(h, k) + k\psi(h, k) \qquad ...(3)$$

where $A = f_x$, $B = f_y$ and ϕ, ψ are function of h, k tending to zero as h, k tend to zero simultaneously.

2. We have seen that a function differentiable at a point is necessarily continuous and possesses partial derivatives thereat. Not only that, we talk of differentiability at a point of a function only when it is continuous and has partial derivatives thereat, for it is only then that it can be expressed in the form of equation (1).

Let a function f and its partial derivatives $f_x,\ f_y$ be continuous at a point (x, y) of its domain of definition, and let

$$\delta f = f(x + \delta x,\ y + \delta y) - f(x,\ y)$$

$$= \{f(x + \delta x,\ y + \delta y) - f(x,\ y + \delta y)\} + \{f(x,\ y + \delta y) - f(x,\ y)\}$$

Using Lagrange's mean value theorem of one variable, we get

$$\delta f = \delta x\ f_x(x + \theta_1 \delta x,\ y + \delta y) + \delta y\, f_y(x,\ y + \theta_2 \delta y)$$

where $0 < \theta_1 < 1,\ 0 < \theta_2 < 1.$

Since $f_x,\ f_y$ are continuous at (x, y) therefore when $(\delta x, \delta y) \to (0, 0)$, we get

$$\delta f = (f_x + \phi)\, \delta x + (f_y + \psi)\, \delta y$$

when ϕ and ψ tend to zero as $(\delta x, \delta y) \to (0, 0)$

$$\therefore \qquad \delta f = f_x\, \delta x + f_y\, \delta y + \delta x \phi + \delta y \psi$$

We now give an example to show that a function may be continuous and possess partial derivatives at a point and still may not be differentiable thereat.

Example 11.12. Let

$$f(x,\ y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & (x,\ y) \neq (0, 0) \\ 0, & (x,\ y) = (0, 0) \end{cases}$$

Solution. Put $x = r\cos\theta,\ y = r\sin\theta,$

$$\therefore \qquad \left|\frac{x^3 - y^3}{x^2 + y^2}\right| = \left| r(\cos^3\theta - \sin^3\theta)\right| \leq 2\left| r\right| = 2\sqrt{x^2 + y^2} < \varepsilon,$$

if

$$x^2 < \frac{\varepsilon^2}{8},\quad y^2 < \frac{\varepsilon^2}{8}$$

or, if

$$\left| x\right| < \frac{\varepsilon}{2\sqrt{2}},\quad \left| y\right| < \frac{\varepsilon}{2\sqrt{2}}$$

$$\therefore \qquad \left|\frac{x^3 - y^3}{x^2 + y^2} - 0\right| < \varepsilon, \text{ when } \left| x\right| < \frac{\varepsilon}{2\sqrt{2}}, \left| y\right| < \frac{\varepsilon}{2\sqrt{2}}$$

$$\Rightarrow \qquad \lim_{(x,\ y)\to(0,\ 0)} \frac{x^3 - y^3}{x^2 + y^2} = 0$$

$$\Rightarrow \qquad \lim_{(x,\ y)\to(0,\ 0)} f(x,\ y) = f(0, 0)$$

Hence, the function is continuous at (0, 0).

Again,

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{h - 0}{h} = 1$$

$$f_y(0, 0) = \lim_{k\to 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k\to 0} \frac{-k}{k} = -1$$

Thus the function possesses partial derivatives at (0, 0).

If the function is differentiable at (0, 0), then by definition

$$df = f(h, k) - f(0, 0) = Ah + Bk + h\phi + k\psi \qquad ...(1)$$

when A and B are constants $(A = f_x(0, 0) = 1, B = f_y(0, 0) = -1)$ and ϕ, ψ tend to zero as $(h, k) \to (0, 0)$.

Putting $h = \rho \cos\theta$, $k = \rho \sin\theta$, and dividing by ρ, we get

$$\cos^3\theta - \sin^3\theta = \cos\theta - \sin\theta + \phi \cos\theta + \psi \sin\theta \qquad ...(2)$$

For arbitrary $\theta = \tan^{-1}(h/k)$, $\rho \to 0$ implies that $(h, k) \to (0, 0)$. Thus we get the limit,

$$\cos^3\theta - \sin^3\theta = \cos\theta - \sin\theta$$

or

$$\cos\theta \sin\theta (\cos\theta - \sin\theta) = 0$$

which is plainly impossible for arbitary θ.

Thus the function is not differentiable at the origin.

Note. The method used to show that the function is not differentiable, can also be used to show that the function is not continuous at (0, 0); for example:

The function f, where

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & x^2 + y^2 \neq 0 \\ 0, & \text{if } x = y = 0 \end{cases}$$

is not differentiable at the origin because it is discontinuous thereat.

Example 11.13. Show that the function f, where

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & \text{if } x^2 + y^2 \neq 0 \\ 0, & \text{if } x = y = 0 \end{cases}$$

is continuous, possesses partial derivative but is not differentiable at the origin.

Solution. As was shown in Example 11.9, f is continuous at the origin. Also it may be easily shown that

$$f_x(0, 0) = 0 = f_y(0, 0)$$

If the function is differentiable at the origin then by definition

$$df = f(h, k) - f(0, 0) = Ah + Bk + h\phi + k\psi \qquad ...(1)$$

where $A = f_x(0, 0) = 0$; $B = f_y(0, 0) = 0$, and ϕ, ψ tend to zero as $(h, k) \to (0, 0)$.

$$\therefore \qquad \frac{hk}{\sqrt{h^2 + k^2}} = h\phi + k\psi \qquad \ldots(2)$$

Putting $k = mh$ and letting $h \to 0$, we get

$$\frac{m}{\sqrt{1 + m^2}} = \lim_{h \to 0} (\phi + m\psi) = 0$$

which is impossible for arbitrary m.

Hence, the function is not differentiable at (0, 0).

Note. If we put $h = r\cos\theta$, $k = r\sin\theta$ in (2) we get

$$\cos\theta \sin\theta = \phi \cos\theta + \psi \sin\theta$$

For arbitrary θ, $r \to 0$ implies $(h, k) \to (0, 0)$.

Thus when $r \to 0$, we get

$$\cos\theta \cdot \sin\theta = 0$$

which is impossible for arbitrary θ. So f is not differentiable at the origin.

EXERCISE

1. Show that the function f, where

$$f(x, y) = \begin{cases} x\sin 1/x + y\sin 1/y, & xy \neq 0 \\ x\sin 1/x, & y = 0,\ x \neq 0 \\ y\sin 1/y, & x = 0,\ y \neq 0 \\ 0, & x = 0 = y \end{cases}$$

   is continuous but not differentiable at the origin.

2. Show that the function $|x| + |y|$ is continuous, but not differentiable at the origin.

3. Discuss the following functions for continuity and differentiability at the origin:

   (*i*) $f(x, y) = \dfrac{xy^2}{x^2 + y^2}$ when $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$

   (*ii*) $f(x, y) = y\sin 1/x$, if $x \neq 0$, $f(0, y) = y$.

11.4.1 A Sufficient Condition for Differentiability

Theorem 11.2. *If* (a, b) *be a point of the domain of definition of a function f such that*

(*i*) f_x *is continuous at* (a, b),

(*ii*) f_y *exists at* (a, b),

then f is differentiable at (a, b).

The condition (*i*) implies that f_x exists in a certain neighbourhood $(a - \delta, a + \delta; b - \delta, b + \delta)$ of (a, b). Let $(a + h, b + k)$ be a point of this neighbourhood. Thus

$$df = f(a+h, b+k) - f(a, b)$$
$$= f(a+h, b+k) - f(a, b+k) + f(a, b+k) - f(a, b) \quad ...(1)$$

Since f_x exists in $(a-\delta, a+\delta; b-\delta, b+\delta)$, applying Lagrange's mean value theorem, we get

$$f(a+h, b+k) - f(a, b+k) = hf_x(a+\theta h, b+k) \quad ...(2)$$

where $0 < \theta < 1$, and depends on h and k.

Again, since f_x is continuous at (a, b), therefore

$$\lim_{(h, k)\to(0, 0)} f_x(a+\theta h, b+k) = f_x(a, b)$$

so that we can write

$$f_x(a+\theta h, b+k) = f_x(a, b) + \phi(h, k) \quad ...(3)$$

where $\phi(h, k) \to 0$ as $(h, k) \to (0, 0)$.

Again, since by condition (ii), $f_y(a, b)$ exists, therefore

$$\lim_{k\to 0} \frac{f(a, b+k) - f(a, b)}{k} = f_y(a, b)$$

so that we can write

$$\frac{f(a, b+k) - f(a, b)}{k} = f_y(a, b) + \psi(k) \quad ...(4)$$

where $\psi(k) \to 0$ as $k \to 0$.

$\therefore$ from (1), (2), (3) and (4), we get

$$df = hf_x(a, b) + kf_y(a, b) + h\phi(h, k) + k\psi(k)$$

$\Rightarrow$ f is differentiable at (a, b).

Note. In a similar way it can be shown that f is differentiable at (a, b), if f_x exists and f_y is continuous at (a, b). In fact, one of the partial derivatives is to be continuous and the other merely to exist at the point.

Remark. We have shown that the condition of existence of one partial derivative and the continuity of the other is sufficient to ensure that the function is differentiable but with the help of an example (Example I below) we now show that the condition of continuity is not necessary so that function may be differentiable even though none of the partial derivatives is continuous. However, if the function is not differentiable at a point, the partial derivatives cannot be continuous thereat (Example II).

Example I. Consider the function

$$f(x, y) = \begin{cases} x^2 \sin 1/x + y^2 \sin 1/y, & \text{if } xy \neq 0 \\ x^2 \sin 1/x, & \text{if } x \neq 0 \text{ and } y = 0 \\ y^2 \sin 1/y, & \text{if } x = 0 \text{ and } y \neq 0 \\ 0, & \text{if } x = y = 0 \end{cases}$$

The partial derivatives,

$$f_x(x, y) = \begin{cases} 2x \sin 1/x - \cos 1/x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$f_y(x, y) = \begin{cases} 2y \sin 1/y - \cos 1/y, & \text{if } y \neq 0 \\ 0, & \text{if } y = 0 \end{cases}$$

are discontinuous at the origin, so that both the partial derivatives exist at the origin, but none is continuous thereat.

Let us show that the function is differentiable at the origin.

Here $f(h, k) - f(0, 0) = h^2 \sin 1/h + k^2 \sin 1/k$

$$= 0h + 0k + h(h \sin 1/h) + k(k \sin 1/k)$$

Now $(h \sin 1/h)$ and $(k \sin 1/k)$ both tend to zero when $(h, k) \to (0, 0)$ so that f is differentiable at the origin.

Example II. Prove that the function

$$f(x, y) = \sqrt{|xy|}$$

is not differentiable at the point (0, 0), but that f_x and f_y both exist at the origin and have the value 0. Hence deduce that these two partial derivatives are continuous except at the origin.

Solution. Now at (0, 0),

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

$$f_y(0, 0) = \lim_{k \to 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k \to 0} \frac{0}{k} = 0$$

If the function is differentiable at (0, 0), then by definition

$$f(h, k) - f(0, 0) = 0h + 0k + h\phi + k\psi$$

where ϕ and ψ are functions of h, k and tend to zero as $(h, k) \to (0, 0)$.

Putting $h = \rho \cos\theta$, $k = \rho \sin\theta$ and dividing by ρ, we get

$$\left| \cos\theta \sin\theta \right|^{1/2} = \phi \cos\theta + \psi \sin\theta$$

Now for arbitrary θ, $\rho \to 0$ implies that $(h, k) \to (0, 0)$.

Taking the limit as $\rho \to 0$, we get

$$\left| \cos\theta \sin\theta \right|^{1/2} = 0,$$

which is impossible for all arbitrary θ.

Hence, the function is not differentiable at (0, 0) and consequently the partial derivatives f_x, f_y cannot be continuous at (0, 0), for otherwise the function would be differentiable thereat,

Let us now see that it is actually so.

For $(x, y) \neq (0, 0)$

$$f_x(x, y) = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{|x+h||y|} - \sqrt{|x||y|}}{h}$$

$$= \lim_{h\to 0} \sqrt{|y|}\, \frac{|x+h| - |x|}{h\left[\sqrt{|x+h|} + \sqrt{|x|}\right]}.$$

Now as $h \to 0$, we can take $x + h > 0$, *i.e.*, $|x+h| = x + h$, when $x > 0$ and $x + h < 0$ or $|x+h| = -(x+h)$, when $x < 0$.

$\therefore$
$$f_x(x, y) = \begin{cases} \dfrac{1}{2}\sqrt{\dfrac{|y|}{|x|}}, & \text{when } x > 0 \\ -\dfrac{1}{2}\sqrt{\dfrac{|y|}{|x|}}, & \text{when } x < 0 \end{cases}$$

Similarly,

$$f_y(x, y) = \begin{cases} \dfrac{1}{2}\sqrt{\dfrac{|x|}{|y|}}, & \text{when } y > 0 \\ -\dfrac{1}{2}\sqrt{\dfrac{|x|}{|y|}}, & \text{when } y < 0 \end{cases}$$

which are, obviously, not continuous at the origin.

Example 11.14. Show that the function f, where

$$f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & \text{if } x^2 + y^2 \neq 0 \\ 0, & \text{if } x = y = 0 \end{cases}$$

is differentiable at the origin.

Solution. It may be easily shown that

$$f_x(0, 0) = 0 = f_y(0, 0)$$

Also when $x^2 + y^2 \neq 0$,

$$|f_x| = \frac{|x^4 y + 4x^2 y^3 - y^5|}{(x^2 + y^2)^2} \leq \frac{6(x^2 + y^2)^{5/2}}{(x^2 + y^2)^2} = 6(x^2 + y^2)^{1/2}$$

Evidently

$$\lim_{(x,y)\to(0,0)} f_x(x, y) = 0 = f_x(0, 0)$$

Thus f_x is continuous at (0, 0) and $f_y(0, 0)$ exists,

$\Rightarrow f$ is differentiable at (0, 0).

11.4.2 Algebra of Differentiable Functions

If f, g are two functions differentiable at (a, b), then $f \pm g$, fg are differentiable at (a, b); f/g is differentiable at (a, b), if $g(a, b) \neq 0$, and

$$d(f \pm g) = df \pm dg$$
$$d(fg) = gdf + fdg$$
$$d(f/g) = (gdf + fdg)/g^2.$$

11.5 PARTIAL DERIVATIVES OF HIGHER ORDER

If a function f has partial derivatives of the first order at each point (x, y) of a certain region, then f_x, f_y are themselves function of x, y and may also possess partial derivatives. These are called *second order partial derivatives of f* and are denoted by

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} = f_x^2$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy} = f_y^2$$
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}$$

In a similar manner higher order partial derivatives are defined. For example $\dfrac{\partial^3 f}{\partial x \partial x \partial y} = f_{xxy}$ and so on.

The second order partial derivatives at a particular point (a, b) are often denoted by

$$\left[\frac{\partial^2 f}{\partial x^2}\right]_{(a,b)}, \frac{\partial^2 f(a, b)}{\partial x^2}, f_{xx}(a, b) \text{ or } f_x^2(a, b)$$

$$\left[\frac{\partial^2 f}{\partial x \partial y}\right]_{(a,b)}, \frac{\partial^2 f(a, b)}{\partial x \partial y} \text{ or } f_{xy}(a, b)$$

and so on.

Thus

$$f_{xx}(a, b) = \lim_{h\to 0} \frac{f_x(a + h, b) - f_x(a, b)}{h},$$

$$f_{xy}(a, b) = \lim_{h\to 0} \frac{f_y(a + h, b) - f_y(a, b)}{h},$$

$$f_{yx}(a, b) = \lim_{k\to 0} \frac{f_x(a, b + k) - f_x(a, b)}{k},$$

$$f_{yy}(a, b) = \lim_{k\to 0} \frac{f_y(a, b + k) - f_y(a, b)}{k},$$

in case the limits exist.

11.5.1 Change in the Order of Partial Derivation

In most of the cases that occur in practice, a partial derivative has the same value in whatever order the different operations are performed. Thus, for example, it is usually found that

$$f_{xy} = f_{yx}, f_{xyx} = f_{xxy}, f_{xyxy} = f_{xxyy}$$

and one is often tempted to believe that it is always so. But it is not the case and there is no *a priori* reason why they should be equal. Let us now see why f_{xy} may be different from f_{yx} at some point (a, b) of the region.

Now

$$f_{xy}(a, b) = \lim_{h\to 0} \frac{f_y(a + h, b) - f_y(a, b)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\lim_{k\to 0} \frac{f(a + h, b + k) - f(a + h, b)}{k} - \lim_{k\to 0} \frac{f(a, b + k) - f(a, b)}{k}\right]$$

$$= \lim_{h\to 0} \lim_{k\to 0} \frac{f(a + h, b + k) - f(a + h, b) - f(a, b + k) + f(a, b)}{hk} = \lim_{h\to 0} \lim_{k\to 0} \frac{\phi(h, k)}{hk}$$

where $\phi(h, k) = f(a + h, b + k) - f(a + h, b) - f(a, b + k) + f(a, b)$.

Similarly,

$$f_{yx}(a, b) = \lim_{k\to 0} \lim_{h\to 0} \frac{\phi(h, k)}{hk}$$

Thus we see that $f_{xy}(a, b)$ and $f_{yx}(a, b)$ are the repeated limits of the same expression taken in different orders. There is therefore no *a priori* reason why they should always be equal.

Let us consider an example to show that f_{xy} may be different from f_{yx}.

Example 11.15. Let

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}, (x, y) \neq (0, 0), f(0, 0) = 0, \text{ then}$$

show that at the origin $f_{xy} \neq f_{yx}$.

Solution. Now

$$f_{xy}(0, 0) = \lim_{h\to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h}$$

$$f_y(0, 0) = \lim_{k\to 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k\to 0} \frac{0}{k} = 0$$

$$f_y(h, 0) = \lim_{k\to 0} \frac{f(h, k) - f(h, 0)}{k} = \lim_{k\to 0} \frac{hk(h^2 - k^2)}{k \cdot (h^2 + k^2)} = h$$

$\therefore$ $$f_{xy} = \lim_{h\to 0} \frac{h - 0}{h} = 1$$

Again

$$f_{yx}(0, 0) = \lim_{k\to 0} \frac{f_x(0, k) - f_x(0, 0)}{k}$$

But

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = 0$$

$$f_x(0, k) = \lim_{h\to 0} \frac{f(h, k) - f(0, k)}{h} = \lim_{h\to 0} \frac{hk(h^2 - k^2)}{h(h^2 + k^2)} = -k$$

$\therefore$ $$f_{yx}(0, 0) = \lim_{k\to 0} \frac{-k - 0}{k} = -1$$

$\therefore$ $$f_{xy}(0, 0) \neq f_{yx}(0, 0).$$

Example 11.16. Examine the equality of f_{xy} and f_{yx}, where

$$f(x, y) = x^3 y + e^{xy^2}.$$

Solution. Now

$$f_y = x^3 + 2xye^{xy^2}$$

$\therefore$ $$f_{xy} = 3x^2 + 2ye^{xy^2} + 2xy^3 e^{xy^2}$$

Again

$$f_x = 3x^2 y + y^2 e^{xy^2}$$

$$f_{yx} = 3x^2 + 2ye^{xy^2} + 2xy^3 e^{xy^2}$$

$\Rightarrow$ $$f_{xy} = f_{yx}.$$

EXERCISE

1. Verify that $f_{xy} = f_{yx}$ for the functions:

   (a) $\dfrac{2x - y}{x + y}$, (b) $x \tan xy$, (c) $\cosh (y + \cos x)$, (d) x^y

   indicating possible exceptional points and investigate these points.

2. Show that $z = \log \{(x - a)^2 + (y - b)^2\}$, satisfies $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0,$ except at (a, b).

3. Show that $z = x \cos (y/x) + \tan (y/x)$ satisfies $x^2 z_{xx} + 2xyz_{xy} + y^2 z_{yy} = 0$, except at points for which $x = 0$.

4. Prove that $f_{xy} \neq f_{yx}$ at the origin for the function:

   $$f(x, y) = x^2 \tan^{-1} (y/x) - y^2 \tan^{-1} (x/y),\ x \neq 0,\ y \neq 0$$
   $$f(x, y) = 0, \text{ elsewhere.}$$

5. If $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$, show that

   $$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

6. Examine for the change in the order of derivation at the origin for the functions:

   (i) $f(x, y) = e^x (\cos y + x \sin y)$

   (ii) $f(x, y) = \sqrt{x^2 + y^2} \sin 2\phi$,

   where $f(0, 0) = 0$ and $\phi = \tan^{-1} (y/x)$,

   (iii) $f(x, y) = \left| x^2 - y^2 \right|$.

7. Examine the equality of $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$ for the function:

   $$f(x, y) = (x^2 + y^2) \tan^{-1} (y/x),\ \ x \neq 0,\ f(0, y) = \pi y^2/2.$$

8. Given $u = e^x \cos y + e^y \sin z$, find all second partial derivatives and verify that

   $$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}; \frac{\partial^2 u}{\partial x \partial z} = \frac{\partial^2 u}{\partial z \partial x}; \frac{\partial^2 u}{\partial y \partial z} = \frac{\partial^2 u}{\partial z \partial y}.$$

11.5.2 Sufficient Conditions for the Equality of f_{xy} and f_{yx}

As was said earlier there is no *a priori* reason why f_{xy} and f_{yx} should always be equal. We now give two theorems the object of which is to set out precisely under what conditions it is safe to assume that $f_{xy} = f_{yx}$ at a point, *i.e.*, **sufficient conditions** for the equality of f_{xy} and f_{yx}.

Theorem 11.3. (***Young's theorem***). *If f_x and f_y are both differentiable at a point (a, b) of the domain of definition of a function f, then*

$$f_{xy}(a, b) = f_{yx}(a, b).$$

The differentiability of f_x and f_y at (a, b) implies that they exist in a certain neighbourhood of (a, b) and that all the second order partial derivatives $f_{xx}, f_{xy}, f_{yx}, f_{yy}$, exist at (a, b).

We prove the theorem by taking equal increment h both for x and y and calculating $\phi(h, h)$ in two different ways.

Let $(a + h, b + h)$ be a point of this neighbourhood. Consider

$$\phi(h, h) = f(a + h, b + h) - f(a + h, b) - f(a, b + h) + f(a, b)$$

$$G(x) = f(x, b + h) - f(x, b)$$

so that

$$\phi(h, h) = G(a + h) - G(a) \qquad ...(1)$$

Since f_x exists in a neighbourhood of (a, b), the function $G(x)$ is derivable in $]a, a + h[$ and therefore by Lagrange's mean value theorem, we get from (1),

$$\phi(h, h) = hG'(a + \theta h), \quad 0 < \theta < 1$$

$$= h\{f_x(a + \theta h, b + h) - f_x(a + \theta h, b)\} \qquad ...(2)$$

Again, since f_x is differentiable at (a, b), we have

$$f_x(a + \theta h, b + h) - f_x(a, b) = \theta h f_{xx}(a, b) + h f_{yx}(a, b) + \theta h \phi_1(h, h) + h\psi_1(h, h) \qquad ...(3)$$

and

$$f_x(a + \theta h, b) - f_x(a, b) = \theta h f_{xx}(a, b) + \theta h \phi_2(h, h) \qquad ...(4)$$

where ϕ_1, ψ_1, ϕ_2 all tend to zero as $h \to 0$.

From (2), (3), (4), we get

$$\phi(h, h)/h^2 = f_{yx}(a, b) + \theta\phi_1(h, h) + \psi_1(h, h) - \theta\phi_2(h, h) \qquad ...(5)$$

By a similar argument, on considering

$$H(y) = f(a + h, y) - f(a, y)$$

we can show that

$$\phi(h, h)/h^2 = f_{xy}(a, b) + \phi_3(h, h) + \theta'\psi_2(h, h) - \theta'\psi_3(h, h) \qquad ...(6)$$

where ϕ_3, ψ_2, ψ_3 all tend to zero as $h \to 0$.

On taking the limit as $h \to 0$, we obtain from (5) and (6)

$$\lim_{h \to 0} \frac{\phi(h, h)}{h^2} = f_{xy}(a, b) = f_{yx}(a, b).$$

Note. An alternative set of conditions which involves only the existence of one of the second order partial derivatives of f at (a, b) provided we assume also its continuity, is made in the following next theorem.

Theorem 11.4. (***Schwarz's theorem***). *If f_y exists in a certain neighbourhood of a point (a, b) of the domain of definition of a function f, and f_{yx} is continuous at (a, b), then $f_{xy}(a, b)$ exists, and is equal to $f_{yx}(a, b)$.*

Under the given conditions, f_x, f_y, and f_{yx} all exist in a certain neighbourhood of (a, b). Let $(a + h, b + k)$ be a point of this neighbourhood.

Consider

$$\phi(h, k) = f(a + h, b + k) - f(a + h, b) - f(a, b + k) + f(a, b)$$
$$G(x) = f(x, b + k) - f(x, b)$$

so that

$$\phi(h, k) = G(a + h) - G(a) \qquad ...(1)$$

Since f_x exists in a neighbourhood of (a, b), the function $G(x)$ is derivable in $]a, a + h[$, and therefore by Lagrange's mean value theorem, we get from (1)

$$\phi(h, k) = hG'(a + \theta h), \quad 0 < \theta < 1$$
$$= h\{f_x(a + \theta h, b + k) - f_x(a + \theta h, b)\} \qquad ...(2)$$

Again, since f_{yx} exists in a neighbourhood of (a, b), the function f_x is derivable with respect to y in $]b, b + k[$, and therefore by Lagrange's mean value theorem, we get from (2)

$$\phi(h, k) = hkf_{yx}(a + \theta h, b + \theta' k), \quad 0 < \theta' < 1$$

or

$$\frac{1}{h}\left\{\frac{f(a + h, b + k) - f(a + h, b)}{k} - \frac{f(a, b + k) - f(a, b)}{k}\right\} = f_{yx}(a + \theta h, b + \theta' k)$$

Proceeding to limits when $k \to 0$, since f_y and f_{yx} exist in a neighbourhood of (a, b), we get

$$\frac{f_y(a + h, b) - f_y(a, b)}{h} = \lim_{k \to 0} f_{yx}(a + \theta h, b + \theta' k)$$

Again, taking limits as $h \to 0$, since f_{yx} is continuous at (a, b), we get

$$f_{xy}(a, b) = \lim_{h \to 0} \lim_{k \to 0} f_{yx}(a + \theta h, b + \theta' k) = f_{yx}(a, b)$$

Notes 1. If f_{xy} and f_{yx} are both continuous *at* (a, b), then $f_{xy}(a, b) = f_{yx}(a, b)$, for the assumption of continuity of both these derivatives is a wider assumption than those required for proving either Theorem 11.3 or Theorem 11.4.

2. If the conditions of Young's or Schwarz's theorem are satisfied then $f_{xy} = f_{yx}$ at a point (a, b). But if the conditions are not satisfied, we cannot draw any conclusion regarding the equality of f_{xy} and f_{yx} they may or may not be equal (see examples 11.17 and 11.18). Thus the conditions are *sufficient* but *not necessary*.

Example 11.17. Show that for the function

$$f(x, y) = \begin{cases} \dfrac{x^2 y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

$f_{xy}(0, 0) = f_{yx}(0, 0)$, even though the conditions of Schwarz's theorem and also of Young's theorem are not satisfied.

Solution. Now

$$f_x(0, 0) = \lim_{x \to 0} \frac{f(x, 0) - f(0, 0)}{x} = 0$$

Similarly, $f_y(0, 0) = 0$.

Also, for $(x, y) \neq (0, 0)$,

$$f_x(x, y) = \frac{(x^2 + y^2) \cdot 2xy^2 - x^2y^2 \cdot 2x}{(x^2 + y^2)^2} = \frac{2xy^4}{(x^2 + y^2)^2}$$

$$f_y(x, y) = \frac{2x^4y}{(x^2 + y^2)^2}$$

Again

$$f_{yx}(0, 0) = \lim_{y \to 0} \frac{f_x(0, y) - f_x(0, 0)}{y} = 0$$

and $$f_{xy}(0, 0) = 0, \text{ so that } f_{xy}(0, 0) = f_{yx}(0, 0)$$

For $(x, y) \neq (0, 0)$, we have

$$f_{yx}(x, y) = \frac{8xy^3(x^2 + y^2)^2 - 2xy^4 \cdot 4y(x^2 + y^2)}{(x^2 + y^2)^4} = \frac{8x^3y^3}{(x^2 + y^2)^3}$$

and it may be easily shown (by putting $y = mx$) that

$$\lim_{(x, y) \to (0, 0)} f_{yx}(x, y) \neq 0 = f_{yx}(0, 0)$$

so that f_{yx} is not continuous at (0, 0), *i.e.*, the conditions of Schwarz's theorem are not satisfied.

Let us now show that the conditions of Young's theorem are also not satisfied.

Now

$$f_{xx}(0, 0) = \lim_{x \to 0} \frac{f_x(x, 0) - f_x(0, 0)}{x} = 0$$

Also f_x is differentiable at (0, 0) if

$$f_x(h, k) - f_x(0, 0) = f_{xx}(0, 0) \cdot h + f_{yx}(0, 0) \cdot k + h\phi + k\psi$$

or

$$\frac{2hk^4}{(h^2 + k^2)^2} = h\phi + k\psi$$

where ϕ, ψ tend to zero as $(h, k) \to (0, 0)$.

Putting $h = \rho \cos\theta$ and $k = \rho \sin\theta$, and dividing by ρ, we get

$$2 \cos\theta \sin^4\theta = \cos\theta \cdot \phi + \sin\theta . \psi$$

and $(h, k) \to (0, 0)$ is same thing as $\rho \to 0$ and θ is arbitrary. Thus, proceeding to limits, we get

$$2 \cos\theta \sin^4\theta = 0$$

which is impossible for arbitrary θ.

$\Rightarrow$ f_x is not differentiable at (0, 0).

Similarly, it may be shown that f_y is not differentiable at (0, 0).

Thus the conditions of Young's theorem are also not satisfied but, as shown above,

$$f_{xy}(0, 0) = f_{yx}(0, 0).$$

Example 11.18. Show that the function

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}, \quad (x, y) \neq (0, 0)$$

$$f(0, 0) = 0$$

does not satisfy the conditions of Schwarz's theorem and

$$f_{xy}(0, 0) \neq f_{yx}(0, 0).$$

Solution. It may be shown, as in example 11.15, that

$$f_{xy}(0, 0) = 1, \; f_{yx}(0, 0) = -1$$

so that

$$f_{xy}(0, 0) \neq f_{yx}(0, 0).$$

Now, for $(x, y) \neq (0, 0)$ we have

$$f_x(x, y) = \frac{(x^2 + y^2)y(3x^2 - y^2) - 2x^2y(x^2 - y^2)}{(x^2 + y^2)^2} = \frac{y\{x^4 + 4x^2y^2 - y^4\}}{(x^2 + y^2)^2}$$

$$\therefore \qquad f_{yx}(x, y) = \frac{(x^2 + y^2)^2\{x^4 + 12x^2y^2 - 5y^4\} - 4y^2(x^2 + y^2)\{x^4 + 4x^2y^2 - y^4\}}{(x^2 + y^2)^4}$$

$$= \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2 + y^2)^3}$$

By putting $y = mx$ or $x = r\cos\theta, \; y = r\sin\theta,$ it may be shown that

$$\lim_{(x, y) \to (0, 0)} f_{yx}(x, y) \neq -1 = f_{yx}(0, 0).$$

Thus f_{yx} is not continuous at (0, 0).

It may similarly be shown that f_{xy} is also not continuous at (0, 0).

Thus the conditions of Schwarz's theorem are not satisfied.

11.6 DIFFERENTIALS OF HIGHER ORDER

Let $z = f(x, y)$ be a function of two independent variables x and y, defined in a domain N and let it be differentiable at a point (x, y) of the domain. The first differential of z at (x, y), denoted by dz is given by

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \qquad ...(1)$$

If dx and dy are regarded as constants and if $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are differentiable at (x, y) then dz is a function of x and y and is itself differentiable at (x, y). The differential of dz, called the *second differential* of z, is denoted by d^2z and is calculated in the same way as the first.

$$\therefore \quad d^2z = d(dz) = d\left(\frac{\partial z}{\partial x}\right)dx + d\left(\frac{\partial z}{\partial y}\right)dy \qquad ...(2)$$

Replacing z by $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in (1), we get

$$d\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}dx + \frac{\partial^2 z}{\partial y \partial x}dy$$

$$d\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}dx + \frac{\partial^2 z}{\partial y^2}dy$$

Also by Young's theorem, since $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are differentiable, we have

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

$$\therefore \quad d^2z = \frac{\partial^2 z}{\partial x^2}dx^2 + 2\frac{\partial^2 z}{\partial x \partial y}dx\,dy + \frac{\partial^2 z}{\partial y^2}dy^2 \qquad ...(3)$$

where, of course, $dx^2 = dx \cdot dx = (dx)^2$, $dy^2 = (dy)^2$

In abbreviated notation, it may be written as

$$d^2z = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^2 z \qquad ...(4)$$

Again d^2z is differentiable at (x, y) if all the second order partial derivatives $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$ are differentiable at (x, y). This condition also ensures the legitimacy of inverting the order of the partial derivatives with respect to x and with respect to y, and so

$$d^3z = \frac{\partial^3 z}{\partial x^3}dx^3 + 3\frac{\partial^3 z}{\partial x^2 \partial y}dx^2\,dy + 3\frac{\partial^3 z}{\partial x \partial y^2}dx\,dy^2 + \frac{\partial^3 z}{\partial y^3}\partial y^3 = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^3 z. \qquad ...(5)$$

Proceeding in the manner, we can define the successive differentials d^4z, d^5z,... Thus the differential of nth order, d^nz exists if $d^{n-1}z$ is differentiable, which implies that all the partial derivatives of the $(n-1)$th order are differentiable. This condition also ensures the legitimacy of inverting the order of the partial derivatives with respect to x and with respect to y in the partial derivatives of order n. Thus it may be shown by Mathematical induction that

$$d^n z = \frac{\partial^n z}{\partial x^n} dx^n + n \frac{\partial^n z}{\partial x^{n-1} \partial y} dx^{n-1} dy + \frac{n(n-1)}{2!} \frac{\partial^n z}{\partial x^{n-2} \partial y^2} dx^{n-2} dy^2 + ... + \frac{\partial^n z}{\partial y^n} dy^n$$

$$= \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy \right)^n z.$$

Note. In the above discussion, x and y are *Independent Variables* and so dx and dy may be treated as constants. The reason for this being so is that the differentials of independent variables are the arbitrary increments of these variables, $dx = \delta x$, $dy = \delta y$.

11.7 FUNCTIONS OF FUNCTIONS

So far we have consider functions of the form

$$z = f(x, y, ...)$$

where the variables x, y,... are the independent variables. We now consider functions

$$z = f(x, y, ...)$$

where x, y,... are not independent variables, but are themselves functions of other independent variables u, v, ..., so that

$$x = g(u, v, ...) \text{ and } y = h(u, v, ...),$$

To fix the ideas, we consider only two variables x and y as functions of two independent variables u and v. The method of proof is, however, general.

Theorem 11.5. *If $z = f(x, y)$ is a differentiable function of x, y and $x = g(u, v,)$, $y = h(u, v,)$ are themselves differentiable functions of the independent variables u, v, then z is a differentiable function of u, v and*

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

just as though x, y were the independent variables.

Let (u, v), $(u + \delta u, v + \delta v)$ be two neighbouring points of the domain of definition of x and y, and (x, y), $(x + \delta x, y + \delta y)$ the corresponding points of the domain of definition of z, so that

$$\delta x = g(u + \delta u, v + \delta v) - g(u, v)$$

$$\delta y = h(u + \delta u, v + \delta v) - h(u, v)$$

The differentiability, and hence the continuity of g and h imply that

$$\delta x \to 0, \delta y \to 0, \text{ as } (\delta u, \delta v) \to (0, 0)$$

Again, since g and h are differentiable function of u and v,

$$\delta x = g_u \delta u + g_v \delta v + \phi_1 \delta u + \psi_1 \delta v \quad ...(1)$$

$$\delta y = h_u \delta u + h_v \delta v + \phi_2 \delta u + \psi_2 \delta v, \quad ...(2)$$

where $\phi_1, \phi_2, \psi_1, \psi_2$ are functions of $\delta u, \delta v,$ and tend to zero as,

$$(\delta u, \delta v) \to (0, 0). \qquad ...(3)$$

Also, $dx = g_u\, du + g_v\, dv$, $dy = h_u\, du + h_v\, dv$.

Also, since f is a differentiable function of x, y, we have

$$\delta z = f_x\, \delta x + f_y\, \delta y + \phi_3\, \delta x + \psi_3\, \delta y, \qquad ...(4)$$

where ϕ_3, ψ_3 are functions of $\delta x, \delta y,$ and tend to zero as, $(\delta x, \delta y) \to (0, 0)$.

From (1), (2), (3), we get

$$\delta z = (f_x g_u + f_y h_u)\, \delta u + (f_x g_v + f_y h_v)\, \delta v + F_1\, \delta u + F_2\, \delta v$$

where
$$F_1 = f_x \phi_1 + f_y \phi_2 + \phi_3 g_u + \phi_3 \phi_1 + \psi_3 h_u + \psi_3 \phi_2$$

$$F_2 = f_x \psi_1 + f_y \psi_2 + \phi_3 g_v + \phi_3 \psi_1 + \psi_3 h_v + \psi_3 \psi_2$$

Since, the coefficients F_1 and F_2 of $\delta u, \delta v$ tend to zero as $(\delta u, \delta v) \to (0, 0)$, therefore z is a *differentiable function of* u, v and

$$\begin{aligned} dz &= (f_x g_u + f_y h_u)\, du + (f_x g_v + f_y h_v)\, dv \\ &= f_x(g_u\, du + g_v\, dv) + f_y(h_u\, du + h_v\, dv) \\ &= f_x\, dx + f_y\, dy \qquad \text{[Using (4)]} \end{aligned}$$

$$\therefore \qquad dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy. \qquad ...(5)$$

Remark. The theorem establishes a fact of fundamental importance that *the first differential of a function is expressed always by the same formula, whether the variables concerned are independent or whether they are themselves functions of other independent variables.*

Note. The differential dz is sometimes referred to as the *total differential*.

11.7.1 Differentials of Higher Order of a Function of Functions

If $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are differentiable functions of x, y so that they are also differentiable functions of u, v, and dx, dy are differentiable functions of u, v, then from the preceding theorem, we have

$$d^2 z = d(dz) = d\left(\frac{\partial z}{\partial x}\right) dx + \frac{\partial z}{\partial x} d^2 x + d\left(\frac{\partial z}{\partial y}\right) dy + \frac{\partial z}{\partial y} d^2 y,$$

and on comparison with (2) and (3) of § 11.6, we see that

$$d^2 z = \frac{\partial^2 z}{\partial x^2} dx^2 + 2\frac{\partial^2 z}{\partial x \partial y} dx dy + \frac{\partial^2 z}{\partial y^2} dy^2 + \frac{\partial z}{\partial x} d^2 x + \frac{\partial z}{\partial y} d^2 y \qquad ...(1)$$

$$= \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy\right)^2 z + \frac{\partial z}{\partial x} d^2 x + \frac{\partial z}{\partial x} d^2 y. \qquad ...(2)$$

The differentials of higher orders can be written in the same manner, but their formation becomes more and more complicated and lengthy, and no simple general formula for $d^n z$ can be given.

The introduction of more than two *intermediary variables*[*] causes no fresh difficulty. Thus, when $z = f(x_1, x_2, x_3)$ and x_1, x_2, x_3 are not the independent variables,

$$d^2 z = \left(\frac{\partial}{\partial x_1} dx_1 + \frac{\partial}{\partial x_2} dx_2 + \frac{\partial}{\partial x_3} dx_3 \right)^2 z + \frac{\partial z}{\partial x_1} d^2 x_1 + \frac{\partial z}{\partial x_2} d^2 x_2 + \frac{\partial z}{\partial x_3} d^2 x_3.$$

Note. If x, y are linear functions of independent variables u and v, *i.e.*, x and y are of the form $x = a + bu + cv$, $y = a' + b'u + c'v$ then dx and dy are constants, and so d^2x, d^2y and all higher differentials of x and y are zero, and therefore

$$d^n z = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy \right)^n z,$$

the form being same as for independent x and y.

11.7.2 The Derivation of Composite Functions (*The Chain Rule*)

From the preceding theorem we deduce two important results:

I. *If*

(*i*) *x, y be differentiable functions of a single variable, and*

(*ii*) *z is differentiable function of x and y,*

then z possesses continuous derivative with respect to t, and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Because of (*i*),

$$dx = \frac{dx}{dt} \cdot dt, \text{ and } dy = \frac{dy}{dt} \cdot dt$$

Since z is a differentiable function of x and y, and x, y are differentiable functions of t, we deduce from § 7, that z is a differentiable function of t.

$$\therefore \qquad dz = \frac{dz}{dt} \cdot dt \qquad \text{...(1)}$$

Also
$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = \frac{\partial z}{\partial x}\frac{dx}{dt} dt + \frac{\partial z}{\partial y}\frac{dy}{dt} dt \qquad \text{...(2)}$$

From (1) and (2)

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \qquad \text{...(3)}$$

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[*] Variables like x, y which are functions of independent variables u, v are called *intermediary variables*.

Again, because of conditions (*i*) and (*ii*) $\frac{dz}{dt}$ is a continuous function of t.

Corollary. If $z = f(x, y)$ possesses nth order partial derivatives, and x, y are linear functions of a single variable t, *i.e.*, $x = a + ht$, $y = b + kt$, where a, b, h, k are constants, then

$$\frac{d^n z}{dt^n} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^n z$$

Now

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = h\frac{\partial z}{\partial x} + k\frac{\partial z}{\partial y} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right) z \qquad ...(1)$$

Replacing z by $\left(h\frac{\partial z}{\partial x} + k\frac{\partial z}{\partial y} \right)$ in (1), we get

$$\frac{d^2 z}{dt^2} = \frac{d}{dt}\left(\frac{dz}{dt} \right) = h\frac{\partial}{\partial x}\left(h\frac{\partial z}{\partial x} + k\frac{\partial z}{\partial y} \right) + k\frac{\partial}{\partial y}\left(h\frac{\partial z}{\partial x} + k\frac{\partial z}{\partial y} \right)$$

$$= h^2\frac{\partial^2 z}{\partial x^2} + 2hk\frac{\partial^2 z}{\partial x \partial y} + k^2\frac{\partial^2 z}{\partial y^2} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^2 z$$

By induction, we may obtain the required expression for $\frac{d^n z}{dt^n}$.

II. *If*

(*i*) *x, y are differentiable functions of two independent variables u and v, and*

(*ii*) *z is a differentiable function of x and y,*

then z possesses continuous partial derivatives with respect to u and v, and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \text{ and } \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Because of (*i*)

$$\left.\begin{aligned} dx &= \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv \\ dy &= \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv \end{aligned}\right\} \qquad ...(1)$$

Since z is a differentiable function of x and y and x, y are differentiable functions of u and v, we deduce from § 7, that z is a differentiable function of u, and v, and

$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv \qquad ...(2)$$

Also $$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right)$$

$$= \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\right)du + \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\right)dv \qquad ...(3)$$

Hence, from (2) and (3), we get

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \text{ and } \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Again, because of conditions (*i*) and (*ii*) we see that $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ are continuous functions of u, v.

Note. In (1) when x is a function of a single variable t, we have $dx = \frac{dx}{dt}dt$, so that the derivative $\frac{dx}{dt}$ appears as the coefficient of a differential and that is precisely the reason why the derivative is also called the *differential coefficient*.

Example 11.19. If $z = e^{xy^2}$, $x = t\cos t$, $y = t\sin t$, compute $\frac{dz}{dt}$ at $t = \frac{\pi}{2}$.

Solution. $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (y^2e^{xy^2})(\cos t - t\sin t) + (2xye^{xy^2})(\sin t + t\cos t)$$

At $$t = \frac{\pi}{2},\ x = 0,\ y = \frac{\pi}{2}.$$

$\therefore$ $$\left[\frac{dz}{dt}\right]_{t=\pi/2} = \frac{\pi^2}{4}\left(-\frac{\pi}{2}\right) = -\frac{\pi^3}{8}$$

Example 11.20. If $z = x^3 - xy + y^3$, $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial z}{\partial r}, \frac{\partial z}{\partial \theta}$.

Solution. $$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = (3x^2 - y)\cos\theta + (3y^2 - x)\sin\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = (3x^2 - y)(-r\sin\theta) + (3y^2 - x)\,r\cos\theta$$

Example 11.21. Show that $z = f(x^2\,y)$, where f is differentiable, satisfies

$$x\left(\frac{\partial z}{\partial x}\right) = 2y\left(\frac{\partial z}{\partial y}\right).$$

Solution. Let $x^2y = u$, so that $z = f(u)$. Thus

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = f'(u)\cdot 2xy$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = f'(u) \cdot x^2$$

$$\therefore \qquad x\frac{\partial z}{\partial x} = f'(u)\ 2x^2 y = 2y\frac{\partial z}{\partial y}$$

Aliter. $dz = f'(u)\,du = f'(x^2 y)\ (2xy\ dx + x^2\ dy)$

Also $dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial x}dy,$

then $\dfrac{\partial z}{\partial x} = 2xyf'(x^2 y),\ \dfrac{\partial z}{\partial y} = x^2 f'(x^2 y)$

The result now follows as above.

Example 11.22. If for all values of the parameter λ, and for some constant n, $F(\lambda x, \lambda y) = \lambda^n F(x, y)$ (F is then called a *homogeneous function* of degree n), identically where F is assumed differentiable, prove that, $x\dfrac{\partial F}{\partial x} + y\dfrac{\partial F}{\partial y} = nF$. Hence show that, for $F(x, y) = x^4 y^2 \sin^{-1}\dfrac{y}{x}$,

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = nF.$$

Solution. Let $\lambda x = u$, $\lambda y = v$. Then

$$F(u, v) = \lambda^n F(x, y) \qquad ...(1)$$

The derivative w.r.t. λ of the left side of (1) is

$$\frac{\partial F}{\partial \lambda} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial \lambda} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial \lambda} = x\frac{\partial F}{\partial u} + y\frac{\partial F}{\partial v}$$

The derivative w.r.t. λ of the right side of (1) is $n\lambda^{n-1} F(x, y)$.

Then $$x\frac{\partial F}{\partial u} + y\frac{\partial F}{\partial v} = n\lambda^{n-1}F$$

The result follows for $\lambda = 1$, for then $u = x$, $v = y$.

Again, since $F(\lambda x, \lambda y) = (\lambda x)^4 (\lambda y)^2 \sin^{-1} y/x = \lambda^6 F(x, y)$, the result follows for $n = 6$.

That it is so, can also be shown by direct differentiation.

Example 11.23. If z is given as a function of two independent variables x and y, change the variables so that x becomes the function, and z and y the independent variables, and express the first and second order partial derivatives of x with respect to z and y in terms of the derivatives of z with respect to x and y.

Solution. When x and y are independent variables and z the dependent, a usual notation (which will be often employed) is

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial x \partial y} = s, \frac{\partial^2 z}{\partial y^2} = t$$

We know

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = p\,dx + q\,dy \qquad ...(1)$$

Again, when z and y are independent and x the function,

$$dx = \frac{\partial x}{\partial z} dz + \frac{\partial x}{\partial y} dy \qquad ...(2)$$

From (1),

$$dx = \frac{1}{p} dz - \frac{q}{p} dy \qquad ...(3)$$

Comparing the coefficients in equations (2) and (3), we get

$$\frac{\partial x}{\partial z} = \frac{1}{p}, \frac{\partial x}{\partial y} = -\frac{q}{p} \qquad ...(4)$$

Taking the differential of the first, we have

$$d\left(\frac{\partial x}{\partial z}\right) = d\left(\frac{1}{p}\right)$$

or

$$\frac{\partial^2 x}{\partial z^2} dz + \frac{\partial^2 x}{\partial y \partial z} dy = -\frac{1}{p^2} dp \qquad ...(5)$$

But p is a function of x and y.

$\therefore$ $$dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy = r\,dx + s\,dy$$

Hence, from (5),

$$\frac{\partial^2 x}{\partial z^2} dz + \frac{\partial^2 x}{\partial y \partial z} dy = -\frac{1}{p^2}(r\ dx + s\ dy) = -\frac{r}{p^3} dz + \frac{rq - sp}{p^3} dy \ \text{ [using (3)]}$$

In this equation we have only the differentials of independent variables and can therefore equate the coefficients of dz and dy; hence

$$\frac{\partial^2 x}{\partial z^2} = -\frac{r}{p^3}, \frac{\partial^2 x}{\partial y \partial z} = \frac{rq - sp}{p^3} \qquad ...(6)$$

In the same way, the second equation of (4) gives

$$\frac{\partial^2 x}{\partial z \partial y} dz + \frac{\partial^2 x}{\partial y^2} dy = -\frac{p\,dq - q\,dp}{p^2} \qquad ...(7)$$

But $dq = \frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial y} dy = s\,dx + t\,dy,$ substituting, as before for dp, dq and using (3), we have from (7),

$$\frac{\partial^2 x}{\partial z \partial y} dz + \frac{\partial^2 x}{\partial y^2} dy = \frac{rq - sp}{p^3} dz + \frac{2pqs - tp^2 - rq^2}{p^3} dy,$$

and, therefore

$$\frac{\partial^2 x}{\partial y^2} = \frac{2pqs - tp^2 - rq^2}{p^3}.$$

The value of $\partial^2 x/\partial z \partial y$ being the same as $\partial^2 x/\partial y \partial z$.

Note. $$\frac{\partial^2 x}{\partial z^2} \cdot \frac{\partial^2 x}{\partial y^2} - \left(\frac{\partial^2 x}{\partial z \partial y} \right)^2 = \frac{rt - s^2}{p^4}$$

11.8 CHANGE OF VARIABLES

In problems involving change of variables it is frequently required to transform a particular expression involving a combination of derivatives with respect to a set of variables, in terms of derivatives with respect to another set of variables. A general method, illustrating the principles involved, is given below, but it can often be modified so as to reduce the algebraic work.

We shall consider derivatives up to second order only. The higher derivatives may be obtained by exactly the same method; fortunately they are not often required. The algebra of the transformation is tedious but the method seems simple.

Problem. *If z is a function $f(x, y)$ of the independent variables x, y, and if x, y are changed to new independent variables u, v by the substitutions $x = \phi(u, v)$, $y = \psi(u, v)$, it is required to express the derivatives of z with respect to x, y in terms of u, v and the derivatives of z with respect to u, v.*

It is understood that f, ϕ, ψ are differentiable (or possesses continuous partial derivatives) with respect to the corresponding variables.

But rule II of § 11.7.2, we have

$$\left.\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \end{aligned}\right\} \qquad ...(1)$$

Solving these for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, we get

$$\left.\begin{aligned}\frac{\partial z}{\partial x} &= A\frac{\partial z}{\partial u} + B\frac{\partial z}{\partial v}\\ \frac{\partial z}{\partial y} &= C\frac{\partial z}{\partial u} + D\frac{\partial z}{\partial v}\end{aligned}\right\} \qquad ...(2)$$

where $$A = \frac{\partial y}{\partial v}\bigg| J,\ B = -\frac{\partial y}{\partial u}\bigg| J,\ C = -\frac{\partial x}{\partial v}\bigg| J,\ D = \frac{\partial x}{\partial u}\bigg| J,$$

$$J = \frac{\partial(x,\ y)}{\partial(u,\ v)} = \begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix},\ \text{called the } \textit{Jacobian},$$

are functions of u and v. (Refer to § 2 of Chapter 16, for properties of Jacobians.)

Thus

$$\frac{\partial z}{\partial x} = -\frac{\partial(y,\ z)}{\partial(u,\ v)}\Big/\frac{\partial(x,\ y)}{\partial(u,\ v)} \text{ and } \frac{\partial z}{\partial y} = -\frac{\partial(z,\ x)}{\partial(u,\ v)}\Big/\frac{\partial(x,\ y)}{\partial(u,\ v)}$$

Equation (2) expresses $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in terms of A, B, C, D, $\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}$ which are all functions of u, v, and not contain x, y explicitly.

From (2),

$$\frac{\partial}{\partial x} = A\frac{\partial}{\partial u} + B\frac{\partial}{\partial v},\ \frac{\partial}{\partial y} = C\frac{\partial}{\partial u} + D\frac{\partial}{\partial v}$$

Replacing z by $\frac{\partial z}{\partial x}$ in (2), we get

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(A\frac{\partial z}{\partial u} + B\frac{\partial z}{\partial v}\right) = \left(A\frac{\partial}{\partial u} + B\frac{\partial}{\partial v}\right)\left(A\frac{\partial z}{\partial u} + B\frac{\partial z}{\partial v}\right)$$

$$= A\frac{\partial}{\partial u}\left(A\frac{\partial z}{\partial u} + B\frac{\partial z}{\partial v}\right) + B\frac{\partial}{\partial v}\left(A\frac{\partial z}{\partial u} + B\frac{\partial z}{\partial v}\right)$$

$$= A^2\frac{\partial^2 z}{\partial u^2} + 2AB\frac{\partial^2 z}{\partial u\partial v} + B^2\frac{\partial^2 z}{\partial v^2} + \left(A\frac{\partial A}{\partial u} + B\frac{\partial A}{\partial v}\right)\frac{\partial z}{\partial u} + \left(A\frac{\partial B}{\partial u} + B\frac{\partial B}{\partial v}\right)\frac{\partial z}{\partial v}$$

The values of $\frac{\partial^2 z}{\partial y^2}$ and $\frac{\partial^2 z}{\partial x\partial y}$ may be found in the same way.

Remark. The expression for $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ can also be found as follows:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}, \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \quad ...(3)$$

To obtain $\frac{\partial u}{\partial x}$, differentiate $x = \phi(u, v)$ and $y = \psi(u, v)$ with respect to x,

$$\therefore \quad 1 = \frac{\partial x}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial x}, \text{ and } 0 = \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x}$$

and these give

$$\frac{\partial u}{\partial x} = A, \frac{\partial v}{\partial x} = B \quad ...(4)$$

Differentiating $x = \phi(u, v),\ y = \psi(u, v)$ with respect to y, we get

$$\frac{\partial u}{\partial y} = C, \frac{\partial v}{\partial y} = D \quad ...(5)$$

Equation (3) now gives the required result.

Note. In the 'change of variables' the variables x, y which are functions of u, v are called the *Intermediate variables*, while u, v are independent variables.

Example 11.24. If $u = F(x, y, z)$, and $z = f(x, y)$, find a formula for $\partial^2 u/\partial x^2$ in terms of the derivatives of F and the derivatives of z.

Solution. In the expression for F we consider x, y, z as intermediate variables, while in the expression for f we consider x and y as independent variables.

Now $$\frac{\partial u}{\partial x} = F_x \frac{\partial x}{\partial x} + F_y \frac{\partial y}{\partial x} + F_z \frac{\partial z}{\partial x},$$

but since x and y are independent, $\frac{\partial y}{\partial x} = 0.$

Also, $$\frac{\partial x}{\partial x} = 1$$

$$\therefore \quad \frac{\partial u}{\partial x} = F_x + F_z \frac{\partial z}{\partial x}$$

Differentiating a second time, we get

$$\frac{\partial^2 u}{\partial x^2} = F_{xx}\frac{\partial x}{\partial x} + F_{yx}\frac{\partial y}{\partial x} + F_{zx}\frac{\partial z}{\partial x} + \frac{\partial z}{\partial x}\left(F_{xz}\frac{\partial x}{\partial x} + F_{yz}\frac{\partial y}{\partial x} + F_{zz}\frac{\partial z}{\partial x}\right) + F_z\frac{\partial^2 z}{\partial x^2}$$

$$= F_{x^2} + 2F_{zx}\frac{\partial z}{\partial x} + F_{z^2}\left(\frac{\partial z}{\partial x}\right)^2 + F_z\frac{\partial^2 z}{\partial x^2}$$

of course, F and f are supposed to be differentiable.

Example 11.25. Show that $f(xy,\ z - 2x) = 0$ satisfies, under suitable conditions, the equation $x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y} = 2x$. What are these conditions?

Solution. Let $u = xy$, $v = z - 2x$; then $f(u, v) = 0$, and

$$df = f_u\, du + f_v\, dv = f_u(x\, dy + y\, dx) + f_v(dz - 2\, dx) = 0$$

Taking z as dependent variable and x and y as independent variables, we have

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$\therefore \qquad df = f_u(x\,dy + y\,dx) + f_v\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy - 2\,dx\right) = 0$$

or

$$\left\{yf_u + f_v\left(\frac{\partial z}{\partial x} - 2\right)\right\}dx + \left\{xf_u + f_v\frac{\partial z}{\partial y}\right\}dy = 0$$

But, since x and y are independent, we have

$$yf_u + f_v\left(\frac{\partial z}{\partial x} - 2\right) = 0, \text{ and } xf_u + f_v\frac{\partial z}{\partial y} = 0.$$

Finding f_u from one equation and putting in the other, we get

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2x, \text{ provided } f_v \neq 0$$

Thus the result holds when f is differentiable and $f_v \neq 0$ (and then $f_u \neq 0$).

Example 11.26. Prove that, by the transformations $u = x - ct$, $v = x + ct$, the partial differential equation

$$\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2} \text{ reduces to } \frac{\partial^2 z}{\partial u \partial v} = 0$$

Solution. In this problem we consider z as a function of u and v which are linear functions of the independent variables x and t.

Now by § 11.7.1,

$$d^2z = \frac{\partial^2 z}{\partial u^2}du^2 + 2\frac{\partial^2 z}{\partial u \partial v}du\,dv + \frac{\partial^2 z}{\partial v^2}dv^2 + \frac{\partial z}{\partial u}d^2u + \frac{\partial z}{\partial v}d^2v$$

But $u = x - ct$, and $v = x + ct$

$\therefore$ $du = dx - cdt$, $dv = dx + cdt$

$d^2u = 0$, $d^2v = 0$

$$\therefore \qquad d^2z = \frac{\partial^2 z}{\partial u^2}(dx - cdt)^2 + 2\frac{\partial^2 z}{\partial u \partial v}(dx - cdt)\,(dx + cdt) + \frac{\partial^2 z}{\partial v^2}(dx + cdt)^2$$

$$= \left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right) dx^2 + B\ dx\ dt + c^2 \left\{ \frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right\} dt^2 \qquad ...(1)$$

the coefficient of $dx\ dt$ is written as B since its actual value is not required for this problem.

Again regarding z as a function of independent variables x and t, we have (by § 6)

$$d^2 z = \frac{\partial^2 z}{\partial x^2} dx^2 + 2\frac{\partial^2 z}{\partial x \partial t} dx\, dt + \frac{\partial^2 z}{\partial t^2} dt^2 \qquad ...(2)$$

From (1) and (2) by comparing the coefficients

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \qquad ...(3)$$

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left\{ \frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right\} \qquad ...(4)$$

$\therefore$ $$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2} \Rightarrow \frac{\partial^2 z}{\partial u \partial v} = 0$$

Note. $\dfrac{\partial^2 z}{\partial t^2} + c^2 \dfrac{\partial^2 z}{\partial x^2}$ reduces to $2c^2 \left(\dfrac{\partial^2 z}{\partial u^2} + \dfrac{\partial^2 z}{\partial v^2} \right)$.

Example 11.27. Prove that $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2}$ is invariant for change of rectangular axes.

Solution. The change of origin does not affect the expression, for then $x = a + x'$, $y = b + y'$, and $dx = dx'$, $dy = dy'$, and all the partial derivatives of V remain of the same form.

Let the axes turn through an angle α, so that

$x = x' \cos\alpha - y' \sin\alpha$, and $y = x' \sin\alpha + y' \cos\alpha$,

where α is a constant.

We have

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy$$

$$= \frac{\partial V}{\partial x} (dx' \cos\alpha - dy' \sin\alpha) + \frac{\partial V}{\partial y} (dx' \sin\alpha + dy' \cos\alpha)$$

$$= \left(\frac{\partial V}{\partial x} \cos\alpha + \frac{\partial V}{\partial y} \sin\alpha \right) dx' + \left(-\frac{\partial V}{\partial x} \sin\alpha + \frac{\partial V}{\partial y} \sin\alpha \right) dy'$$

Also $$dV = \frac{\partial V}{\partial x'} dx' + \frac{\partial V}{\partial y'} dy'$$

$$\therefore \qquad \frac{\partial V}{\partial x'} = \frac{\partial V}{\partial x}\cos\alpha + \frac{\partial V}{\partial y}\sin\alpha, \text{ and } \frac{\partial V}{\partial y'} = -\frac{\partial V}{\partial x}\sin\alpha + \frac{\partial V}{\partial y}\cos\alpha$$

These give

$$\left.\begin{aligned} \frac{\partial V}{\partial x} &= \frac{\partial V}{\partial x'}\cos\alpha - \frac{\partial V}{\partial y'}\sin\alpha \\ \frac{\partial V}{\partial y} &= \frac{\partial V}{\partial x'}\sin\alpha + \frac{\partial V}{\partial y'}\cos\alpha \end{aligned}\right\} \qquad ...(1)$$

$$\therefore \qquad \frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) = \left(\frac{\partial}{\partial x'}\cos\alpha - \frac{\partial}{\partial y'}\sin\alpha\right)\left(\frac{\partial V}{\partial x'}\cos\alpha - \frac{\partial V}{\partial y'}\sin\alpha\right)$$

$$= \frac{\partial^2 V}{\partial x'^2}\cos^2\alpha - 2\frac{\partial^2 V}{\partial x'\partial y'}\sin\alpha\cos\alpha + \frac{\partial^2 V}{\partial y'^2}\sin^2\alpha$$

$$\frac{\partial^2 V}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) = \left(\frac{\partial}{\partial x'}\sin\alpha + \frac{\partial}{\partial y'}\cos\alpha\right)\left(\frac{\partial V}{\partial x'}\sin\alpha + \frac{\partial V}{\partial y'}\cos\alpha\right)$$

$$= \frac{\partial^2 V}{\partial x'^2}\sin^2\alpha + 2\frac{\partial^2 V}{\partial x'\partial y'}\sin\alpha\cos\alpha + \frac{\partial^2 V}{\partial y'^2}\cos^2\alpha$$

$$\therefore \qquad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial x'^2} + \frac{\partial^2 V}{\partial y'^2}$$

Thus the expression remains invariant.

Note. If we write $x' = x\cos\alpha + y\sin\alpha$, $y' = -x\sin\alpha + y\cos\alpha$ the procedure is slightly simplified.

Example 11.28. If V is a function of two variables x and y and $x = r\cos\theta$, $y = r\sin\theta$, prove that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r}\frac{\partial V}{\partial r}$$

Solution. We have

$$\frac{\partial V}{\partial r} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial V}{\partial x} + \sin\theta\frac{\partial V}{\partial y}$$

$$\frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial V}{\partial x} + r\cos\theta\frac{\partial V}{\partial y}$$

Solving these, we get

$$\frac{\partial V}{\partial x} = \cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta} \Rightarrow \frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial V}{\partial y} = \sin\theta \frac{\partial V}{\partial r} + \frac{\cos\theta}{r}\frac{\partial V}{\partial y} \Rightarrow \frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

Hence,

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right)$$

$$= \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta \frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}\right)$$

$$= \cos^2\theta \frac{\partial^2 V}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r \partial \theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{2\cos\theta\sin\theta}{r^2}\frac{\partial V}{\partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial V}{\partial r}$$

Similarly,

$$\frac{\partial^2 V}{\partial y^2} = \sin^2\theta \frac{\partial^2 V}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r} \cdot \frac{\partial^2 V}{\partial r \partial \theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 V}{\partial \theta^2} - \frac{2\cos\theta\sin\theta}{r^2}\frac{\partial V}{\partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial V}{\partial r}$$

$$\therefore \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r}\frac{\partial V}{\partial r}$$

Deduction 1.

$$\frac{\partial^2 V}{\partial x \partial y} = \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\sin\theta \frac{\partial V}{\partial r} + \frac{\cos\theta}{r}\frac{\partial V}{\partial \theta}\right)$$

$$= \cos\theta\sin\theta \frac{\partial^2 V}{\partial r^2} + \frac{\cos 2\theta}{r}\frac{\partial^2 V}{\partial r \partial \theta} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial^2 V}{\partial \theta^2} - \frac{\cos 2\theta}{r^2}\frac{\partial V}{\partial \theta} - \frac{\sin\theta\cos\theta}{r}\frac{\partial V}{\partial r}$$

Also

$$x^2 - y^2 = r^2\cos 2\theta, \text{ and } 4xy = 2r^2\sin 2\theta$$

$$\therefore \quad (x^2 - y^2)\left(\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial y^2}\right) + 4xy\frac{\partial^2 V}{\partial x \partial y} = r^2\frac{\partial^2 V}{\partial r^2} - r\frac{\partial V}{\partial r} - \frac{\partial^2 V}{\partial \theta^2}$$

Deduction 2. To show that $\dfrac{\partial^2 \theta}{\partial x \partial y} = -\dfrac{\cos 2\theta}{r^2}$.

Here $x = r\cos\theta$, $y = r\sin\theta$

Differentiating w.r.t. (r, θ are functions of x and y)

$$\left.\begin{aligned} 1 &= \frac{\partial r}{\partial x}\cos\theta - r\sin\theta\frac{\partial \theta}{\partial x} \\ 0 &= \frac{\partial r}{\partial x}\sin\theta - r\cos\theta\frac{\partial \theta}{\partial x} \end{aligned}\right\} \quad \therefore \ \frac{\partial r}{\partial x} = \cos\theta, \quad \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}$$

Differentiating w.r.t. y,

$$\left.\begin{aligned} 0 &= \frac{\partial r}{\partial y}\cos\theta - r\sin\theta\frac{\partial\theta}{\partial y} \\ 1 &= \frac{\partial r}{\partial y}\sin\theta - r\cos\theta\frac{\partial\theta}{\partial y} \end{aligned}\right\} \quad \therefore \quad \frac{\partial r}{\partial y} = \sin\theta, \quad \frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r}$$

Now $\dfrac{\partial^2\theta}{\partial x\partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial\theta}{\partial y}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\cos\theta}{r}\right) = -\dfrac{\cos\theta}{r^2}\dfrac{\partial r}{\partial x} - \dfrac{\sin\theta}{r}\dfrac{\partial\theta}{\partial x} = -\dfrac{\cos 2\theta}{r^2}$

Similarly, it may be shown that

$$\frac{\partial^2\theta}{\partial y\partial x} = -\frac{\cos 2\theta}{r^2}$$

$$\frac{\partial^2 r}{\partial x\partial y} = -\frac{\sin\theta\cos\theta}{r} = \frac{\partial^2 r}{\partial y\partial x}.$$

Note. Using the method of Ded. 2, we get

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial V}{\partial\theta}\frac{\partial\theta}{\partial x} = \cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial\theta}$$

and

$$\frac{\partial V}{\partial y} = \sin\theta\frac{\partial V}{\partial r} + \frac{\cos\theta}{r}\frac{\partial V}{\partial\theta}$$

which are same as in the above example.

Example 11.29. Give that F is a function of x and y and that $x = e^u + e^{-v}$, $y = e^v + e^{-u}$, prove that

$$\frac{\partial^2 F}{\partial u^2} - 2\frac{\partial^2 F}{\partial u\partial v} + \frac{\partial^2 F}{\partial v^2} = x^2\frac{\partial^2 F}{\partial x^2} - 2xy\frac{\partial^2 F}{\partial x\partial y} + y^2\frac{\partial^2 F}{\partial y^2} + x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y}$$

Solution. Here F is a function of the intermediary variables x and y; and u, v, are the independent variables.

Now $x = e^u + e^{-v}$, and $y = e^v + e^{-u}$

$$\therefore \quad dx = e^u\,du - e^{-v}\,dv, \qquad dy = e^v\,dv - e^{-u}\,du$$
$$d^2x = e^u\,du^2 + e^{-v}\,dv^2, \qquad d^2y = e^v\,dv^2 + e^{-u}\,du^2$$

$$d^2F = \frac{\partial^2 F}{\partial x^2}dx^2 + 2\frac{\partial^2 F}{\partial x\partial y}dx\,dy + \frac{\partial^2 F}{\partial y^2}dy^2 + \frac{\partial F}{\partial x}d^2x + \frac{\partial F}{\partial y}d^2y$$

$$= \frac{\partial^2 F}{\partial x^2}(e^u\,du - e^{-v}\,dv)^2 + 2\frac{\partial^2 F}{\partial x\partial y}(e^u\,du - e^{-v}\,dv)\,(e^v\,dv - e^{-u}\,du)$$

$$+ (e^v\,dv - e^{-u}\,du)^2\frac{\partial^2 F}{\partial y^2} + \frac{\partial F}{\partial x}(e^u\,du^2 + e^{-v}\,dv^2) + \frac{\partial F}{\partial y}(e^v\,dv^2 + e^{-u}\,du^2)$$

$$= \left(e^{2u} \frac{\partial^2 F}{\partial x^2} - 2 \frac{\partial^2 F}{\partial x \partial y} + e^{-2u} \frac{\partial^2 F}{\partial y^2} + e^{u} \frac{\partial F}{\partial x} + e^{-u} \frac{\partial F}{\partial y} \right) du^2$$

$$+ 2 \left[-e^{u} e^{-v} \frac{\partial^2 F}{\partial x^2} + (e^{u} e^{v} + e^{-u} e^{-v}) \frac{\partial^2 F}{\partial x \partial y} - e^{v} e^{-u} \frac{\partial F}{\partial y^2} \right] du\ dv$$

$$+ \left(e^{-2v} \frac{\partial^2 F}{\partial x^2} - 2 \frac{\partial^2 F}{\partial x \partial y} + e^{2v} \frac{\partial^2 F}{\partial y^2} + e^{-v} \frac{\partial F}{\partial x} + e^{v} \frac{\partial F}{\partial y} \right) dv^2$$

Also $\quad d^2F = \dfrac{\partial^2 F}{\partial u^2} du^2 + 2 \dfrac{\partial^2 F}{\partial u \partial v} du\ dv + \dfrac{\partial^2 F}{\partial v^2} dv^2$

Comparing the coefficients, we get

$$\frac{\partial^2 F}{\partial u^2} = e^{2u} \frac{\partial^2 F}{\partial x^2} - 2 \frac{\partial^2 F}{\partial x \partial y} + e^{-2u} \frac{\partial^2 F}{\partial y^2} + e^{u} \frac{\partial F}{\partial x} + e^{-u} \frac{\partial F}{\partial y}$$

$$\frac{\partial^2 F}{\partial u \partial v} = - e^{-u} e^{-v} \frac{\partial^2 F}{\partial x^2} + (e^{u} e^{v} + e^{-u} e^{-v}) \frac{\partial^2 F}{\partial x \partial y} - e^{v} e^{-u} \frac{\partial^2 F}{\partial y^2}$$

$$\frac{\partial^2 F}{\partial v^2} = e^{-2v} \frac{\partial^2 F}{\partial x^2} - 2 \frac{\partial^2 F}{\partial x \partial y} + e^{2v} \frac{\partial^2 F}{\partial y^2} + e^{-v} \frac{\partial F}{\partial x} + e^{v} \frac{\partial F}{\partial y}$$

$\therefore \quad$ $$\frac{\partial^2 F}{\partial u^2} - 2 \frac{\partial^2 F}{\partial u \partial v} + \frac{\partial^2 F}{\partial x^2} = x^2 \frac{\partial^2 F}{\partial x^2} - 2xy \frac{\partial^2 F}{\partial x \partial y} + y^2 \frac{\partial^2 F}{\partial y^2} + x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y}$$

Remark. The four examples 11.26 to 11.29 give different methods for solving problems of this type. However any one method could be used to solve all such problems. The reader is advised to try.

EXERCISE

1. If $x = u \cos\alpha - v \sin\alpha$ and $y = u \sin\alpha + v \cos\alpha$, where α is a constant, show that

$$\left(\frac{\partial V}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial y} \right)^2 = \left(\frac{\partial V}{\partial u} \right)^2 + \left(\frac{\partial V}{\partial v} \right)^2$$

2. If $2axz + 2byz + cz^2 = k$, $ax + by + cz = R$, prove that

$$R^3 \frac{\partial^2 z}{\partial x^2} = a^2 k,\ R^3 \frac{\partial^2 z}{\partial x \partial y} = abk,\ R^3 \frac{\partial^2 z}{\partial y^2} = b^2 k\ .$$

3. If $z^3 + 3\ (ax + by)z = c^3$, prove that

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = \frac{2z(ax + by)^3}{(ax + by + z^2)^3}\ .$$

4. If $ax^3 + by^3 + cz^3 + 3hxyz = k$, show that

$$(hxy + cz^2)^3 \frac{\partial^2 z}{\partial x \partial y} = hk(hxy - cz^2) - 2(abc + h^3)x^2 y^2 z\,.$$

5. Give that $u = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$, verify that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 z}{\partial z^2} = 0$$

6. If $z = a \tan^{-1}(y/x)$, show that
   (*i*) $(1 + q^2)r - 2pqs + (1 + p^2)t = 0$
   (*ii*) $(rt - s^2)/(1 + p^2 + q^2)^2 = -a^2/(x^2 + y^2 + a^2)^2$
   where p, q, r, s, t have their usual meaning as in Example 11.23.
7. If $u = y^2 + z^{-2}$, $v = z^2 + x^{-2}$, $w = x^2 + y^{-2}$ and if V is a function of x, y, z, prove that

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} + 2\left(u\frac{\partial V}{\partial u} + v\frac{\partial V}{\partial v} + w\frac{\partial V}{\partial w}\right) = 4\left(y^2\frac{\partial V}{\partial u} + z^2\frac{\partial V}{\partial v} + x^2\frac{\partial V}{\partial w}\right)$$

8. If $z = f[(ny - mz)/(nx - lz)]$, prove that

$$(nx - lz)\frac{\partial z}{\partial x} + (ny - mz)\frac{\partial z}{\partial y} = 0$$

9. If $u = f(x + 2y) + g(x - 2y)$, show that

$$4\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}\,.$$

10. If $u = \phi(x + at) + \psi(x - at)$, show that

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

Prove that, if $y = x + at$, $z = x - at$, the equation becomes $\dfrac{\partial^2 u}{\partial y \partial z} = 0.$

11. Given that $u = F(x, y, z)$ and $z = f(x, y)$, find $\dfrac{\partial^2 u}{\partial y^2}$ and $\dfrac{\partial^2 u}{\partial y \partial x}$ in terms of the derivatives of F and f (as in Example 11.24).
12. If $V = F(x, y)$ and $x = e^u \cos t$, $y = e^u \sin t$, show that

$$\frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial t^2} = e^{2u}\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right).$$

13. If $V = F(x, y)$ and $x = \dfrac{1}{2}u(e^v + e^{-v})$, $y = \dfrac{1}{2}u(e^v - e^{-v})$, show that

$$V_{xx} - V_{yy} = V_{uu} + \frac{1}{u}V_u + \frac{1}{u^2}V_{vv}$$

14. If z is a function of u and v, and $u = x^2 - y^2 - 2xy$, $v = y$, prove that the equation

$$(x + y)\frac{\partial z}{\partial x} + (x - y)\frac{\partial z}{\partial y} = 0 \text{ is equivalent to } \frac{\partial z}{\partial v} = 0.$$

15. If $x = u + v$, $y = uv$ and z is a function of x and y, prove that

$$\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} = (x^2 - 4y)\frac{\partial^2 z}{\partial y^2} - 2\frac{\partial z}{\partial y}$$

16. If $x = r\cos\theta,\ y = r\sin\theta,$ prove that the equation

$$xy\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right) - (x^2 - y^2)\frac{\partial^2 u}{\partial x \partial y} = 0$$

becomes

$$r\frac{\partial^2 u}{\partial r \partial \theta} - \frac{\partial u}{\partial \theta} = 0.$$

17. If $x = c\cosh u\cos v$, $y = c\sinh u\sin v$, and F is a function of x and y, show that

$$F_{uu} + F_{vv} = \frac{1}{2}c^2(\cosh 2u - \cos 2v)\,(F_{xx} + F_{yy}).$$

18. If V is a function of u, v, and $u = x^2 - y^2$, $v = 2xy$, prove that

$$4(u^2 + v^2)\frac{\partial^2 V}{\partial u \partial v} + 2u\frac{\partial V}{\partial v} + 2v\frac{\partial V}{\partial u} = xy\left(\frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial y^2}\right) + \frac{1}{2}(x^2 - y^2)\frac{\partial^2 V}{\partial x \partial y}.$$

19. Given that f is a function of x and y and that $x = u^2 v$, $y = uv^2$, prove that

$$2x^2 f_x^2 + 2y^2 f_y^2 + 5xyf_{xy} = uvf_{uv} - \frac{2}{3}\left(uf_u + vf_v\right)$$

20. Prove that, if in the equation

$$\frac{\partial^2 z}{\partial x^2} + 2xy^2\frac{\partial z}{\partial x} + 2(y - y^3)\frac{\partial z}{\partial y} + x^2 y^2 z^2 = 0,$$

the variables, x, y are changed to u, v, where $x = uv$, $y = \frac{1}{v}$, the new equation is obtained by writing u for x and v for y, then z is the same function of u, v as of x, y.

21. If the variables x, y in the equation

$$(x^2 + y)^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) + 4xy\frac{\partial^2 z}{\partial x \partial y} + 2x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y} = 0$$

are changed to u, v, where $2x = e^u + e^v$, $2y = e^u - e^v$, show that the new equation is

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = 0.$$

22. If $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta,$ prove that

$$\frac{\partial r}{\partial x} = \sin\theta\cos\phi,\ \frac{\partial r}{\partial y} = \sin\theta\sin\phi,\ \frac{\partial r}{\partial z} = \cos\theta,$$

$$\frac{\partial\theta}{\partial x} = \frac{1}{r}\cos\theta\,\cos\phi,\ \frac{\partial\theta}{\partial y} = \frac{1}{r}\cos\theta\,\sin\phi,\ \frac{\partial\theta}{\partial z} = -\frac{1}{r}\sin\theta,$$

$$\frac{\partial\phi}{\partial x} = -\frac{\sin\phi}{r\,\sin\theta},\ \frac{\partial\phi}{\partial y} = \frac{\cos\phi}{r\,\sin\theta},\ \frac{\partial\phi}{\partial z} = 0.$$

Find also the derivatives of x, y, z with respect to $r,\ \theta,\ \phi$.

23. If u is a function of x, y, z prove, using values in Exercise 11.22, that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial u}{\partial\theta}\right)^2 + \left(\frac{1}{r\,\sin\theta}\frac{\partial u}{\partial\phi}\right)^2$$

11.9 TAYLOR'S THEOREM

If $f(x,\ y)$ *is a function which possesses continuous partial derivatives of order n in any domain of a point* (a, b), *and the domain is large enough to contain a point* $(a + h, b + k)$ *with it, then there exists a positive number* $0 < \theta < 1,$ *such that*

$$f(a + h, b + k) = f(a, b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a, b) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a, b)$$

$$+ \ldots + \frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(a, b) + R_n\,,$$

where $\quad R_n = \dfrac{1}{n!}\left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)^n f(a+\theta h, b + \theta k),\ 0 < \theta < 1.$

Let $x = a + th$, $y = b + tk$, where $0 \le t \le 1$ is a parameter, and

$$f(x,\ y) = f(a + th, b + tk) = \phi(t)$$

Since the partial derivatives of $f(x,\ y)$ of order n are continuous in the domain under consideration, $\phi^n(t)$ is continuous in [0, 1], and also

$$\phi'(t) = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f$$

$$\phi''(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$\phi^{(n)}(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f\,,$$

therefore by Maclaurin's theorem

$$\phi(t) = \phi(0) + t\phi'(0) + \frac{t^2}{2!}\phi''(0) + ... + \frac{t^{n-1}}{(n-1)!}\phi^{(n-1)}(0) + \frac{t^n}{n!}\phi^{(n)}(\theta t),$$

where $0 < \theta < 1$.

Now putting $t = 1$, we get

$$\phi(1) = \phi(0) + \phi'(0) + \frac{1}{2!}\phi''(0) + ... + \frac{1}{(n-1)!}\phi^{(n-1)}(0) + \frac{1}{n!}\phi^{(n)}(0)$$

But $\phi(1) = f(a + h, b + k)$, and $\phi(0) = f(a, b)$

$$\phi'(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a, b),$$

$$\phi''(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a, b),$$

$$\vdots$$

$$\phi'''^{(n)}(\theta) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a + \theta h, b + \theta k)$$

$$\therefore f(a + h, b + k) = f(a, b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a, b)$$

$$+ \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a, b) + ... + \frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(a, b) + R_n$$

where $R_n = \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a + \theta h, b + \theta k),\ 0 < \theta < 1.$

R_n is called the *remainder after n terms*, and the theorem, *Taylor's theorem with remainder* or *Taylor's expansion* about the point (a, b).

If we put $a = b = 0$; $h = x$, $k = y$, we get

$$f(x, y) = f(0, 0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0, 0)$$

$$+ \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0, 0) + ... + \frac{1}{(n-1)!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^{n-1} f(0, 0) + R_n$$

where $R_n = \frac{1}{n!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \right)^n f(\theta x, \theta y),\ 0 < \theta < 1,$ is called the **Maclaurin's theorem** or *Maclaurin's expansion.*

It is easy to see that Taylor's theorem can also be put in the form:

$$f(a + h, b + k) = f(a, b) + df(a, b) + \frac{1}{2!}d^2 f(a, b) + ...$$

$$+ \frac{1}{(n-1)!}d^{n-1} f(a, b) + \frac{1}{n!}d^n f(a + \theta h, b + \theta k)$$

The reasoning in the general case of several variables is precisely the same and so the theorem can be easily extended to any number of variables.

11.9.1 The Theorem can be Stated in Still another Form

$$f(x,\ y) = f(a, b) + \left[(x - a)\frac{\partial}{\partial x} + (y - b)\frac{\partial}{\partial y} \right] f(a, b)$$

$$+ \frac{1}{2!}\left[(x - a)\frac{\partial}{\partial x} + (y - b)\frac{\partial}{\partial y} \right]^2 f(a, b) + ...$$

$$+ \frac{1}{(n-1)!}\left[(x - a)\frac{\partial}{\partial x} + (y - b)\frac{\partial}{\partial y} \right]^{n-1} f(a, b) + R_n$$

where $R_n = \frac{1}{n!}\left[(x - a)\frac{\partial}{\partial x} + (y - b)\frac{\partial}{\partial y} \right]^n f(a + (x - a)\theta, b + (y - b)\ \theta),$

$0 < \theta < 1,$ called the Taylor's expansion of $f(x,\ y)$ about the point (a, b) in powers of $x - a$ and $y - b$.

Example 11.30. Expand $x^2y + 3y - 2$ in powers of $x - 1$ and $y + 2$. Let us use Taylor's expansion with $a = 1$, $b = -2$. Then

$f(x,\ y) = x^2 y + 3y - 2,$ $\qquad f(1, -2) = -10$

$f_x(x,\ y) = 2xy,$ $\qquad f_x(1, -2) = -4$

$f_y(x,\ y) = x^2 + 3,$ $\qquad f_y(1, -2) = 4$

$f_{xx}(x,\ y) = 2y,$ $\qquad f_{xx}(1, -2) = -4$

$f_{xy}(x,\ y) = 2x,$ $\qquad f_{xy}(1, -2) = 2$

$f_{yy}(x,\ y) = 0,$ $\qquad f_{yy}(1, -2) = 0$

$$f_{xxx}(x, y) = 0 = f_{yyy}(x, y), \qquad f_{yxx}(1, -2) = 2 = f_{xxy}(1, -2)$$

All higher derivatives are zero.

$$\therefore \quad x^2 y + 3y - 2 = -10 - 4(x-1) + 4(y+2) + \tfrac{1}{2}[-4(x-1)^2 + 4(x-1)(y+2)]$$

$$+ \frac{1}{3!} 3(x-1)^2 (y+2)(2) + 0 = -10 - 4(x-1) + 4(y+2) - 2(x-1)^2$$

$$+ 2(x-1)(y+2) + (x-1)^2 (y+2).$$

Example 11.31. If $f(x, y) = \sqrt{|xy|}$, prove that Taylor's expansion about the point (x, x) is not valid in any domain which includes the origin.

Solution. As was shown earlier in Example II § 4.1,

$$f_x(0, 0) = 0 = f_y(0, 0)$$

$$f_x(x, y) = \begin{cases} \frac{1}{2}\sqrt{|y/x|}, & x > 0 \\ -\frac{1}{2}\sqrt{|y/x|}, & x < 0 \end{cases}$$

$$f_y(x, y) = \begin{cases} \frac{1}{2}\sqrt{|x/y|}, & y > 0 \\ -\frac{1}{2}\sqrt{|x/y|}, & y < 0 \end{cases}$$

$$\therefore \qquad f_x(x, x) = f_y(x, x) = \begin{cases} \frac{1}{2}, & x > 0 \\ -\frac{1}{2}, & x < 0 \end{cases}$$

Now Taylor's expansion about (x, x) for $n = 1$, is

$$f(x + h, x + h) = f(x, x) + h[f_x(x + \theta h, x + \theta h) + f_y(x + \theta h, x + \theta h)]$$

or

$$|x + h| = \begin{cases} |x| + h, & \text{if } x + \theta h > 0 \\ |x| - h, & \text{if } x + \theta h < 0 \\ |x|, & \text{if } x + \theta h = 0 \end{cases} \qquad \ldots(1)$$

If the domain $(x, x; x + h, x + h)$ includes the origin, then x and $x + h$ must be of opposite signs, that is either

$$|x + h| = x + h, \quad |x| = -x$$

or

$$|x + h| = -(x + h), \quad |x| = x$$

But under these conditions none of the inequalities (1) holds. Hence, the expansion is not valid.

EXERCISE

1. Expand $x^4 + x^2y^2 - y^4$ about the point (1, 1) up to terms of the second degree. Find the form of R_2.

2. Find the expansion of sin x sin y about (0, 0) up to and including the terms of the fourth degree in (x, y). Compare the result with that you get by multiplying the series for sin x and sin y.

3. Expand $e^x \tan^{-1} y$ about (1, 1) up to the second degree in $(x - 1)$ and $(y - 1)$.

4. Show that the expansion of sin (xy) in powers of $(x - 1)$ and $(y - \pi/2)$ up to and including second degree terms is

$$1 - \frac{1}{8}\pi^2(x-1)^2 - \frac{1}{2}\pi(x-1)\left(y - \frac{\pi}{2}\right) - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2$$

5. Show that, for $0 < \theta < 1$,

$$\sin x \sin y = xy - \frac{1}{6}[(x^3 + 3xy^2)\cos\theta x \sin\theta y + (y^3 + 3x^2y)\sin\theta x \cos\theta y]$$

6. Prove that the first four terms of the Maclaurin expansion of $e^{ax} \cos by$ are

$$1 + ax + \frac{a^2x^2 - b^2y^2}{2!} + \frac{a^3x^3 - 3ab^2xy^2}{3!}.$$

7. Prove that for $0 < \theta < 1$,

$$e^{ax} \sin by = by + abxy + \frac{1}{6}[(a^3x^3 - 3ab^2xy^2)\sin(b\theta y)$$

$$+ (3a^2bx^2y - b^3y^3)\cos(b\theta y)]\, e^{a\theta x}$$

8. Show that if f, f_x, f_y are all continuous in a domain D of (a, b), and D is large enough to contain the point $(a + h, b + k)$ within it, then for $0 < \theta < 1$,

$$f(a + h, b + k) = f(a, b) + hf_x(a + \theta h, b + \theta k) + kf_y(a + \theta h, b + \theta k).$$

If $f(x, y) = x\sqrt{x^2 + y^2}$, $a = b = -1$, $h = k = 3$, verify that the above conditions are satisfied and find the value of θ.

11.10 EXTREME VALUES: MAXIMA AND MINIMA

The theory of extreme values (maximum or minimum) for functions of one variable was considered in an earlier chapter. We now investigate the theory for explicit functions of more than one variable. That for implicit functions will be discussed in the next chapter.

Let (a, b) be a point of the domain of definition of a function f. Then $f(a, b)$ is an *extreme value* of f, if for every point (x, y), [other than (a, b)] of some neighbourhood of (a, b), the difference

$$f(x, y) - f(a, b) \qquad \ldots(1)$$

keeps the same sign.

The extreme value $f(a, b)$ is called *a maximum* or *a minimum value* according as the sign of (1) is negative or positive.

11.10.1 Necessary Condition

A necessary condition for $f(x, y)$ *have an extreme value at* (a, b) *is that* $f_x(a, b) = 0$, $f_y(a, b) = 0$, *provided these partial derivatives exist*.

If $f(a, b)$ is an extreme value of the function $f(x, y)$ of two variables then it must also be an extreme value of both the functions, $f(x, b)$ and $f(a, y)$ of one variable. But a necessary condition that these have extreme value at $x = a$ and $y = b$ respectively, is

$$f_x(a, b) = 0,\ f_y(a, b) = 0.$$

Notes **1.** The function $f(x, y) = |x| + |y|$ has an extreme value at (0, 0) even though the partial derivatives f_x and f_y do not exist at (0, 0).

2. If $f(x, y) = 0$, if $x = 0$ or $y = 0$, and $f(x, y) = 1$ elsewhere, then both the partial derivatives exist (each equal to zero) at the origin, but $f(0, 0)$ is not an extreme value. Thus, the conditions obtained above are *only necessary and not sufficient*.

3. Point at which $f_x = 0, f_y = 0$ (or $df = 0$) are called *Stationary points*.

11.10.2 Sufficient Conditions for $f(x, y)$ to have an Extreme Value at (a, b)

Let $f_x(a, b) = 0 = f_y(a, b)$. Further, lets us suppose that $f(x, y)$ possesses continuous second order partial derivatives in a certain neighbourhood of (a, b) and that these derivatives at (a, b) viz. $f_{xx}(a, b), f_{xy}(a, b)$, $f_{yy}(a, b)$ are not all zero.

Let $(a + h, b + k)$ be a point of this neighbourhood.

Let us write

$$A = f_{xx}(a, b),\ B = f_{xy}(a, b),\ C = f_{yy}(a, b)$$

By Taylor's theorem, we have for $0 < \theta < 1$,

$$f(a + h, b + k) = f(a, b) + [hf_x(a, b) + kf_y(a, b)] + \frac{1}{2!}[h^2 f_{xx}(a + \theta h, b + \theta k)$$

$$+ 2hkf_{xy}(a + \theta h, b + \theta k) + k^2 f_{yy}(a + \theta h, b + \theta k)]$$

But $f_x(a, b) = 0 = f_y(a, b)$, and

Since the second order partial derivatives are continuous at (a, b), we write

$$f_{xx}(a + \theta h, b + \theta k) - f_{xx}(a, b) = \rho_1$$

$$f_{xy}(a + \theta h, b + \theta k) - f_{xy}(a, b) = \rho_2$$

$$f_{yy}(a + \theta h, b + \theta k) - f_{yy}(a, b) = \rho_3$$

where ρ_1, ρ_2, ρ_3 are function of h and k, and $\to 0$ as $(h, k) \to (0, 0)$.

$$\therefore \qquad f(a + h, b + k) - f(a, b) = \frac{1}{2}[Ah^2 + 2Bhk + Ck^2 + \rho]$$

where $\rho = \rho_1 + \rho_2 + \rho_3 \to 0$ as $(h, k) \to (0, 0)$ and is of unknown sign.

Let $G = Ah^2 + 2Bhk + Ck^2$, so that

$$f(a + h, b + k) - f(a, b) = \frac{1}{2}[G + \rho] \qquad ...(1)$$

There are several cases to consider.

(*i*) G never vanishes and keeps a constant sing. Since $\rho \to 0$ when $(h, k) \to (0, 0)$, therefore ρ is a small number and the sign of $G + \rho$ is same as of G, *i.e.,* $G + \rho$, remains negative or positive according as G is negative or positive. Thus the difference,

$f(a + h, b + k) - f(a, b) \lesseqgtr 0$ according as $G \lesseqgtr 0$

But we know by definition that $f(x, y)$ has a maxima or a minima at (a, b) according as the difference $f(a + h, b + k) - f(a, b)$ is negative or positive for all (h, k) except $(0, 0)$.

Thus $f(a, b)$ will be a maximum or a minimum value according as G is negative or positive.

(*ii*) If G can change sign, since $f(a + h, b + k) - f(a, b)$ and G have the same sign when ρ is small, $f(a, b)$ will not be an extreme value.

(*iii*) If G, without ever changing sign, vanishes for certain values of (h, k), the sign of $f(a + h, b + k) - f(a, b)$ will depend upon ρ, which is of unknown sign, and so no conclusion can be drawn. This is the *doubtful case* and requires further investigation.

Let first $A \neq 0$.

G may be written in the form:

$$G = \frac{(Ah + Bk)^2 + k^2(AC - B^2)}{A}$$

(1) If $AC - B^2 > 0$, the numerator of G is the sum of two positive quantities and it never vanishes except when $k = 0$, $h = 0$, simultaneously, which is not permissible [see (*i*)]. Hence G never vanishes and has the same sign as A.

Thus $f(a, b)$ is a maximum value if $A < 0$, and a minimum value if $A > 0$.

(2) If $AC - B^2 < 0$, the sign of the numerator of G may be positive or negative according as $(Ah + Bk)^2 >$ or $< k^2(B^2 - AC)$, *i.e.* according to the values of (h, k). Hence G does not keep the same sign for all values of (h, k), and therefore $f(a, b)$ is not an extreme value.

(3) If $AC - B^2 = 0$, the numerator of G is a perfect square but may vanish for values of (h, k) for which $Ah + Bk = 0$. Thus G, without changing sign, may vanish for certain values of (h, k).

This is the doubtful case in which the sign of $f(a + h, b + k) - f(a, b)$ depends upon ρ and requires further investigation.

If $A = 0$, then

$$G = 2Bhk + Ck^2 = k(2Bh + Ck)$$

(4) If $A = 0$, $B \neq 0$, G changes sign with k and $(2Bh + Ck)$, and there is no extreme value.

(5) If $A = 0$, $B = 0$, G does not change sign but may vanish when $k = 0$ (without $h = 0$). This is therefore the doubtful case and requires further investigation.

Rule. *$f(a, b)$ is an extreme value of $f(x, y)$, if $f_x(a, b) = 0 = f_y(a, b)$, and*

$$f_{xx}(a, b) \cdot f_{yy}(a, b) - [f_{xy}(a, b)]^2 > 0,$$

and this extreme value is a maximum or a minimum according as $f_{xx}(a, b)$ [or $f_{yy}(a, b)$] is negative or positive.

Further investigation is necessary, if

$$f_{xx}(a, b)\, f_{yy}(a, b) - [f_{xy}(a, b)]^2 = 0$$

Remark. Since at (a, b).

$$df = hf_x(a, b) + kf_y(a, b)$$

and

$$d^2f = h^2f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2f_{yy}(a, b) = Ah^2 + 2Bhk + Ck^2$$

so $f(a, b)$ is an extreme value of $f(x, y)$ if at (a, b), $df = 0$, and d^2f keeps the same sign for all values of $(h, k) \neq (0, 0)$.

Note. Discussion of the doubtful case involves the consideration of terms of higher order than the second in the Taylor expansion of $f(a + h, b + k)$ but this is generally not easy and will not be considered here.

However, it is sometimes possible to decide whether f has a maxima or a minima at (a, b) by algebraic or geometric considerations.

Example 11.32. Find the maxima and minima of the function

$$f(x, y) = x^3 + y^3 - 3x - 12y + 20$$

Solution. We have

$$f_x(x, y) = 3x^2 - 3 = 0, \text{ when } x = \pm 1$$

$$f_y(x, y) = 3y^2 - 12 = 0, \text{ when } y = \pm 2$$

Thus the function has four stationary points:

$$(1, 2), (-1, 2), (1, -2), (-1, -2)$$

Now

$$f_{xx}(x, y) = 6x, f_{xy}(x, y) = 0, f_{yy}(x, y) = 6y$$

At (1, 2)

$$f_{xx} = 6 > 0, \text{ and } f_{xx}f_{yy} - (f_{xy})^2 = 72 > 0$$

Hence (1, 2) is a point of minima of the function.

At (–1, 2)

$$f_{xx} = -6, \text{ and } f_{xx}f_{yy} - (f_{xy})^2 = -72 < 0$$

Hence, the function has neither a maxima nor a minima at (–1, 2).

At (1, –2),

$$f_{xx} = 6, \text{ and } f_{xx}f_{yy} - (f_{xy})^2 = -72 < 0.$$

Hence, the function has neither maximum nor minimum at (1, –2).

At (–1, –2),

$$f_{xx} = -6, \text{ and } f_{xx} f_{yy} - (f_{xy})^2 = 72 > 0$$

Hence the function has a maximum value at (–1, –2).

Note. Stationary points like (–1, 2), (1, –2) which are not extreme points are called the *saddle points*.

Example 11.33. Show that the function

$$f(x, y) = 2x^4 - 3x^2 y + y^2$$

has neither a maximum nor a minimum at (0, 0), where

$$f_{xx} f_{yy} - (f_{xy})^2 = 0$$

Solution. Now

$$f_x(x, y) = 8x^3 - 6xy, \; f_y(x, y) = -3x^2 + 2y$$

$\therefore$ $$f_x(0, 0) = 0 = f_y(0, 0)$$

Also

$$f_{xx}(x, y) = 24x^2 - 6y = 0, \text{ at } (0, 0)$$
$$f_{xy}(x, y) = -6x = 0, \text{ at } (0, 0)$$
$$f_{yy}(x, y) = 2, \text{ at } (0, 0)$$

Thus at (0, 0), $f_{xx}(0, 0) \cdot f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = 0.$

so that it is a *doubtful* case, and so requires further examination.

Again

$$f(x, y) = (x^2 - y)\,(2x^2 - y), \; f(0, 0) = 0$$

or

$$f(x, y) - f(0, 0) = (x^2 - y)(2x^2 - y)$$
$$> 0, \text{ for } y < 0 \text{ or } x^2 > y > 0$$
$$< 0, \text{ for } y > x^2 > \frac{y}{2} > 0$$

Thus $f(x, y) - f(0, 0)$ does not keep the same sign near the origin. Hence, f has neither a maximum nor a minimum value at the origin.

Example 11.34. Show that

$$f(x, y) = y^2 + x^2 y + x^4, \text{ has a minimum at } (0, 0).$$

Solution. It can be easily verified that at the origin.

$$f_x = 0, f_y = 0, f_{xx} = 0, f_{xy} = 0, f_{yy} = 2.$$

Thus at the origin $f_{xx} f_{yy} - (f_{xy})^2 = 0,$ so that it is a *doubtful* case and requires further investigation.

But we can write

$$f(x, y) = \left(y + \frac{1}{2}x^2 \right)^2 + \frac{3}{4}x^4$$

and

$$f(x, y) - f(0, 0) = \left(y + \frac{1}{2}x^2 \right)^2 + \frac{3}{4}x^4$$

which is greater than zero for all values of (x, y). Hence, f has a minimum value at the origin.

EXERCISE

1. Examine the following for extreme values:
   (*i*) $4x^2 - xy + 4y^2 + x^3y + xy^3 - 4$ (*ii*) $x^3y^2(12 - 3x - 4y)$,
   (*iii*) $y^2 + 4xy + 3x^2 + x^3$ (*iv*) $(x^2 + y^2 - 4)^2 - x^2$,
   (*v*) $(x^2 + y^2)\, e^{6x+2x^2}$ (*vi*) $(x - y)^2 (x^2 + y^2 - 2)$.
2. Investigate the maxima and minima of the functions,
   (*i*) $21x - 12x^2 - 2y^2 + x^3 + xy^2$ (*ii*) $2(x - y)^2 - x^4 - y^4$
   (*iii*) $x^2 + 3xy + y^2 + x^3 + y^3$ (*iv*) $x^2 + 4xy + 4y^2 + x^3 + 2x^2y + y^4$
   (*v*) $x^2y^2 - 5x^2 - 8xy - 5y^2$.
3. Show that the function
   $$(y - x)^4 + (x - 2)^4$$
   has a minimum at (2, 2).
4. Prove that the function
   $$f(x, y) = x^2 - 2xy + y^2 + x^4 + y^4$$
   has a minima at the origin.
5. Find and classify the extreme values (if any) of the functions:
   (*i*) $x^2 + y^2 + x + y + xy$, (*ii*) $y^2 - x^3$.
6. A rectangular box, open at the top, is to have a volume of 32 cu ft. What must be the dimensions so that the total surface is a minimum.
7. Show that the function
   $$f(x, y) = x^2 - 3xy^2 + 2y^4$$
   has neither a maximum nor a minimum value at the origin.

   [*Hint:* $f(x, y) = (x - y^2)(x - 2y^2)$].

11.11 FUNCTIONS OF SEVERAL VARIABLES

We conclude the chapter by referring briefly—in fact very briefly, to the functions of several variables.

An ordered set $(a_1, a_2, ..., a_n)$ of n real numbers is called a **point** in a space of n dimensions. The aggregate of points $(x_1, x_2, ..., x_n)$ when $x_1, x_2, ..., x_n$ range over the entire set of real numbers is referred to as the *space of n dimensions*, denoted by $\boldsymbol{R}^n$.

Neighbourhoods. The set of values $x_1, x_2, ..., x_n$ other than $a_1, a_2, ..., a_n$ that satisfy the conditions

$$|x_1 - a_1| < \delta, |x_2 - a_2| < \delta, ..., |x_n - a_n| < \delta$$

where δ is an arbitrarily small positive number, is said to form a *neighbourhood* of the point $(a_1, a_2, ..., a_n)$. The neighbourhood may however be specified in other, though equivalent ways.

The rectangle

$$(a_1 - h_1, a_1 + h_1; a_2 - h_2, a_2 + h_2; ..., a_n - h_n, a_n + h_n)$$

where $h_1, h_2, ..., h_n$ are arbitrarily small positive numbers, is said to be rectangular neighbourhood of $(a_1, a_2, ..., a_n)$.

The points inside the sphere $x_1^2 + x_2^2 + ... + x_n^2 = \delta^2$ may be taken as a neighbourhood of the point (0, 0, ..., 0), called a spherical *nbd*.

Continuity. A function $f(x_1, x_2, ..., x_n)$ is said to be *continuous at a point* $P(a_1, a_2, ..., a_n)$, if to every positive number ε, there corresponds a neighbourhood of P such that for every point $(x_1, x_2, ..., x_n)$ of this neighbourhood

$$\left| f(x_1, x_2, ..., x_n) - f(a_1, a_2, ..., a_n) \right| < \varepsilon$$

A function which is continuous at every point of a region is said to be *continuous in that region*.

Partial Derivatives

The derivative of a function with respect to one variable, when all others are kept constant, is called the *partial derivative* of the function with respect to that variable. Thus the partial derivative of f with respect to x_1 at $(a_1, a_2, ..., a_n)$ is

$$\lim_{h_1 \to 0} \frac{f(a_1 + h_1, a_2, ..., a_n) - f(a_1, a_2, ..., a_n)}{h_1}$$

and is denoted by $f_{x_1}(a_1, a_2, ..., a_n)$

The other partial derivatives of the first, second or higher orders may be defined similarly.

Differentiability. A function f is said to be *differentiable* at $(a_1, a_2, ..., a_n)$, if the change δf in the value of the function, when the point changes from $(a_1, a_2, ..., a_n)$ to $(a_1 + h_1, a_2 + h_2, ..., a_n + h_n)$, can be expressed in the form

$$\delta f = A_1 h_1 + A_2 h_2 + ... + A_n h_n + h_1 \phi_1 + h_2 \phi_2 + ... + h_n \phi_n,$$

where $\phi_1, \phi_2, ..., \phi_n$ are functions of $h_1, h_2, ..., h_n$ and tend to zero as $(h_1, h_2, ..., h_n) \to (0, 0, ..., 0)$.

A function which is differentiable at every point of a region, is said to be differentiable over the region.

The differentials df, d^2f, ..., may now be easily defined as in the case of functions of two variables.

11.11.1 Extreme Values of a Function of *n* Variables

A point $(a_1, a_2, ..., a_n)$ is said to be an *extreme point*, and $f(a_1, a_2, ..., a_n)$ an *extreme value* of a function f, if for every point $(x_1, x_2, ..., x_n)$, other than $(a_1, a_2, ..., a_n)$ of some neighbourhood of $(a_1, a_2, ..., a_n)$, the difference,

$$f(x_1, x_2, ..., x_n) - f(a_1, a_2, ..., a_n) \qquad ...(1)$$

keeps the same sign. The extreme value is a **maximum** or a **minimum value** according as the sign is negative or positive.

If $f(a_1, a_2, ..., a_n)$ is an extreme value of the function f of n variables, then it must also be an extreme value of the function $f(x_1, a_2, ..., a_n)$ of one variable x_1 and therefore the partial derivative $f_{x_1}(a_1, a_2, ..., a_n)$, in case it exists, must be zero. The same is true for all the other variables $x_2, x_3, ..., x_n$.

Thus the **necessary conditions** for $f(a_1, a_2, ..., a_n)$ be an extreme value of the function f are that all the partial derivatives $f_{x_1}, f_{x_2}, ..., f_{x_n}$, in case they exist, vanish at $(a_1, a_2, ..., a_n)$.

Since these are only necessary and not sufficient conditions, therefore points which satisfy these conditions may not be extreme points. A point $(a_1, a_2, ..., a_n)$ is called a **stationary point** if all the first order partial derivatives of the function vanish at that point. Thus the stationary points are determined by solving the following n equations simultaneously.

$$f_{x_1}(x_1, x_2, ..., x_n) = 0$$
$$f_{x_2}(x_1, x_2, ..., x_n) = 0$$
$$...$$
$$f_{x_n}(x_1, x_2, ..., x_n) = 0$$

For a function f of n independent variables $x_1, x_2, ..., x_n$ the condition can be given in a more compact form.

Thus, if $(a_1, a_2, ..., a_n)$ is a stationary point, then

$$df(a_1, a_2, ..., a_n) = 0$$

i.e., the differential of the function vanishes at a stationary point. For, at the stationary point all the partial derivatives vanish and therefore

$$df(a_1, a_2, ..., a_n) = f_{x_1}(a_1, a_2, ..., a_n)\,dx_1 + f_{x_2}(a_1, a_2, ..., a_n)\,dx_2$$
$$+ ... + f_{x_n}(a_1, a_2, ..., a_n)\,dx_n = 0$$

Conversly, when $df = 0$, the coefficients of the differentials $dx_1, dx_2, ..., dx_n$ of independent variables, are separately equal to zero.

Further investigations are necessary to decide whether a stationary point is an extreme point or not, or whether it is a maxima or a minima. We now state a rule (without giving a proof which is beyond the scope of this book) for a function of three variables but is applicable to a function of any number of variables.

The Rule

For a function $f(x, y, z)$ of three independent variables, sufficient conditions for (a, b, c) to be an extreme point are that

(*i*) $df(a, b, c) = f_x dx + f_y dy + f_z dz = 0$, *so that* $f_x = f_y = f_z = 0$, *and*

(*ii*) $d^2 f(a, b, c) = f_{xx}(dx)^2 + f_{yy}(dy)^2 + f_{zz}(dz)^2 + 2f_{xy}\,dx\,dy + 2f_{yz}\,dy\,dz + 2f_{zx}\,dz\,dx$,

keep the same sign for arbitrary values of dx, dy, dz; the extreme point being a maxima or a minima according as the sign of d^2f is negative or positive. The point will be neither a maxima nor a minima if d^2f does not keep the same sign; and requires further investigation, if d^2f keeps the same sign but vanishes at some points of a nbd of (a, b, c).

The conditions that d^2f keeps the same sign may be stated (*without proof*) *in terms of matrices*, *as follows*:

Consider the matrix

$$\begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$$

d^2f will always be positive if and only if the three principal minors

$$f_{xx}, \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}, \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix}$$

are all positive, and d^2f will be always negative if and only if their signs are alternatively negative and positive.

Example 11.35. Show that

$$f(x, y, z) = (x + y + z)^3 - 3(x + y + z) - 24xyz + a^3$$

has a minima at (1, 1, 1) and a maxima at (–1, –1, –1).

Solution. We have

$$f_x = 3(x + y + z)^2 - 24yz - 3$$
$$f_y = 3(x + y + z)^2 - 24zx - 3$$
$$f_z = 3(x + y + z)^2 - 24xy - 3$$

Hence, the stationary points are given by

$$\left.\begin{aligned} (x + y + z)^2 - 8yz - 1 = 0 \\ (x + y + z)^2 - 8zx - 1 = 0 \\ (x + y + z)^2 - 8xy - 1 = 0 \end{aligned}\right\}$$

Subtracting second equation from the first,

$$z(x - y) = 0.$$

Similarly, $x(y - z) = 0$, $y(z - x) = 0$.

Therefore, either $x = 0$, $y = 0$, $z = 0$ or $x = y = z$

Hence stationary points are (1, 1, 1), (–1, –1, –1).

Again, we have

$$f_{xx} = 6(x + y + z) = f_{yy} = f_{zz}$$
$$f_{xy} = 6(x + y + z) - 24z = f_{yx}$$
$$f_{yz} = 6(x + y + z) - 24x = f_{zy}$$
$$f_{zx} = 6(x + y + z) - 24x = f_{xz}$$

At (1, 1, 1),

$$f_{xx} = f_{yy} = f_{zz} = 18, f_{xy} = f_{yz} = f_{zx} = -6$$

$$\therefore \quad d^2f = 18(dx^2 + dy^2 + dz^2) - 12(dx\,dy + dy\,dz + dz\,dx)$$
$$= 6[(dx^2 + dy^2 + dz^2) + (dx - dy)^2 + (dy - dz)^2 + (dz - dx)^2]$$

which is positive for all values of dx, dy, dz and does not vanish for

$$(dx, dy, dz) \neq (0, 0, 0)$$

Thus (1, 1, 1) is a point of minima of the function.

At (–1, –1, –1),

$$f_{xx} = f_{yy} = f_{zz} = -18, f_{xy} = f_{yz} = f_{zx} = 6$$

$$\therefore \quad d^2f = -18(dx^2 + dy^2 + dz^2) + 12(dx\,dy + dy\,dz + dz\,dx)$$
$$= -6[(dx^2 + dy^2 + dz^2) + (dx - dy)^2 + (dy - dz)^2 + (dz - dx)^2]$$

which is negative for all dx, dy, dz and never vanishes. Hence the function has a maximum value at (–1, –1, –1).

Example 11.36. Show that the minimum and the maximum values of

$$f(x, y, z) = (ax + by + cz)\, e^{-\alpha^2x^2 - \beta^2y^2 - \gamma^2z^2} \text{ are}$$

$$-\sqrt{\frac{1}{2}(a^2\alpha^{-2} + b^2\beta^{-2} + c^2\gamma^{-2})/e} \text{ and } \sqrt{\frac{1}{2}(a^2\alpha^{-2} + b^2\beta^{-2} + c^2\gamma^{-2})/e}$$

Solution. We have

$$f_x = [a - 2\alpha^2 x\, \Sigma ax] \exp(-\Sigma\alpha^2x^2)$$
$$f_y = [b - 2\beta^2 y\, \Sigma ax] \exp(-\Sigma\alpha^2x^2)$$
$$f_z = [c - 2\gamma^2 z\, \Sigma ax] \exp(-\Sigma\alpha^2x^2).$$

At the stationary point, since $\exp(-\Sigma\alpha^2x^2) \neq 0$, we have

$$\left.\begin{aligned} a - 2\alpha^2 x\, \Sigma ax = 0 \\ b - 2\beta^2 y\, \Sigma ax = 0 \\ c - 2\gamma^2 z\, \Sigma ax = 0 \end{aligned}\right\} \quad ...(1)$$

$$\left.\begin{aligned} x\,\Sigma ax = a/2\alpha^2 \\ y\,\Sigma ax = b/2\beta^2 \\ z\,\Sigma ax = c/2\gamma^2 \end{aligned}\right\} \begin{aligned} &\text{multiplying by } a, b, c \text{ and adding,} \\ &(\Sigma ax)^2 = \frac{1}{2}\Sigma a^2\alpha^{-2} \end{aligned}$$

$$\therefore \quad \therefore\ \Sigma ax = \pm\sqrt{\frac{1}{2}\Sigma a^2\alpha^{-2}} = \pm k, \text{ say}$$

Hence, from (1), the stationary points are

$$\left(\frac{a}{2\alpha^2k}, \frac{b}{2\beta^2k}, \frac{c}{2\gamma^2k}\right), \left(-\frac{a}{2\alpha^2k}, -\frac{b}{2\beta^2k}, -\frac{c}{2\gamma^2k}\right)$$

Again, we have

$$f_{xx} = -2a^2x[a - 2\alpha^2 x\Sigma ax] \exp(-\Sigma\alpha^2x^2)$$

$$-2\alpha^2[\Sigma ax + ax]\exp(-\Sigma\alpha^2x^2)$$

$$f_{xy} = -2a^2x[b - 2\beta^2 y\Sigma ax]\exp(-\Sigma\alpha^2x^2) - 2\beta^2 ay\exp(-\Sigma\alpha^2x^2)$$

and similar expressions for $f_{yy}, f_{zz}, f_{yz}, f_{zx}$.

At the stationary point $\left(\frac{a}{2\alpha^2k}, \frac{b}{2\beta^2k}, \frac{c}{2\gamma^2k}\right)$, we have $\Sigma\alpha^2x^2 = \frac{1}{2}$,

$$f_{xx} = 0 - 2\alpha^2\left[k + \frac{a^2}{2\alpha^2k}\right]e^{-1/2} = -\frac{1}{\sqrt{e}}\left[2\alpha^2k + \frac{a^2}{k}\right] = -\frac{2\alpha^2k^2 + a^2}{k\sqrt{e}}$$

$$f_{yy} = -\frac{2\beta^2k^2 + b^2}{k\sqrt{e}},\ f_{zz} = -\frac{2\gamma^2k^2 + c^2}{k\sqrt{e}}$$

$$f_{xy} = 0 - \frac{ab}{k\sqrt{e}},\ f_{yz} = -\frac{bc}{k\sqrt{e}},\ f_{zx} = -\frac{ca}{k\sqrt{e}}$$

$\therefore$ $$d^2f = \frac{-1}{k\sqrt{e}}[(2\alpha^2k^2 + a^2)dx^2 + (2\beta^2k^2 + b^2)dy^2 + (2\gamma^2k^2 + c^2)dz^2]$$

$$-\frac{2}{k\sqrt{e}}[ab\,dxdy + bc\,dydz + ca\,dzdx]$$

Now $f_{xx} < 0$

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -\frac{2\alpha^2k^2 + \alpha^2}{k\sqrt{e}} & -\frac{ab}{k\sqrt{e}} \\ -\frac{ab}{k\sqrt{e}} & -\frac{2\beta^2k^2 + b^2}{k\sqrt{e}} \end{vmatrix} = \frac{1}{k^2e}\begin{vmatrix} 2\alpha^2k^2 + a^2 & ab \\ ab & 2\beta^2k^2 + b^2 \end{vmatrix}$$

$$= \frac{2}{e}(2\alpha^2\beta^2k^2 + \alpha^2b^2 + a^2\beta^2) > 0.$$

$$\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = -\frac{1}{k^3e^{3/2}}\begin{vmatrix} 2\alpha^2k^2 + a^2 & ab & ac \\ ab & 2\beta^2k^2 + b^2 & bc \\ ac & bc & 2\gamma^2k^2 + c^2 \end{vmatrix}$$

$$= -\frac{1}{k^3e^{3/2}}[(2\gamma^2k^2 + c^2)2k^2(2\alpha^2\beta^2k^2 + \alpha^2b^2 + a^2\beta^2) + a^2c^2(-2\beta^2k^2) - 2\alpha^2b^2c^2k^2]$$

$$= -\frac{4k}{e^{3/2}}[2\alpha^2\beta^2\gamma^2k^2 + \alpha^2b^2c^2 + \alpha^2\beta^2c^2 + a^2b^2\gamma^2] < 0.$$

Thus the three principal minors have alternatively negative and positive sings and so d^2f is always negative; and hence $(a/2\alpha^2k, b/2\beta^2k, c/2\gamma^2k)$ is a point of maxima, and the maximum value $= ke^{-1/2} = \sqrt{\frac{1}{2}\Sigma a^2\alpha^{-2}/e}$.

At the point $(-a/2\alpha^2k, -b/2\beta^2k, -c/2\gamma^2k)$, it may be shown as above that $\Sigma\alpha^2x^2 = \frac{1}{2}$, and

$$f_{xx} = \frac{2\alpha^2k^2 + a^2}{k\sqrt{e}},\ f_{yy} = \frac{2\beta^2k^2 + b^2}{k\sqrt{e}},\ f_{zz} = \frac{2\gamma^2k^2 + c^2}{k\sqrt{e}}$$

$$f_{xy} = \frac{ab}{k\sqrt{e}},\ f_{yz} = \frac{bc}{k\sqrt{e}},\ f_{zx} = \frac{ca}{k\sqrt{e}}$$

and the three principal minors are of positive sign, so that d^2f is positive, and hence the point in question is a minima and the minimum value of the function $= -ke^{-1/2} = -\sqrt{\frac{1}{2}\Sigma a^2\alpha^{-2}/e}$.

EXERCISE

1. Show that the following functions have a minima at the points indicated
   (*i*) $x^2 + y^2 + z^2 + 2xyz$ at (0, 0, 0)
   (*ii*) $8z + 2x^2 + 3y^2 + 4z^2 - 3xy$ at (0, 0, –1)
   (*iii*) $x^4 + y^4 + z^4 - 4xyz$ at (1, 1, 1).
2. Show that the function
   $$f(x, y, z) = 2xyz - 4zx - 2yz + x^2 + y^2 + z^2 - 2x - 4y + 4z$$
   has 5 stationary points but has a minimum value only at (1, 2, 0).
3. Show that the function
   $$3 \log (x^2 + y^2 + z^2) - 2x^3 - 2y^3 - 2z^3.$$
   has only one extreme value, $\log (3/e^2)$.
4. Show that the function
   $$(y + z)^2 + (z + x)^2 + xyz,$$
   has no maximum or minimum value.
5. If all the letters denote positive numbers, show that the maximum value of
   $$xy(z - h)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2}\right) \text{ is } (2h/5)^5\, ab/c^4.$$

CHAPTER 12

Implicit Functions

12.1 DEFINITION

It is generally assumed that a functional equation $f(x, y) = 0$ determines y as a function of x; but such an equation may not define any such function or it may define one or more than one such function. For example, the equation

$$x^2 + y^2 - 5 = 0$$

determines two functions

$$y = \sqrt{5 - x^2}\,, \text{ and } y = -\sqrt{5 - x^2}\,, \text{ for } x^2 \leq 5,$$

whereas the equation

$$x^2 + y^2 + 5 = 0$$

determines no such function.

Definition. Let $f(x, y)$ be a function of two variables, and $y = \phi(x)$ be a function of x such that, for every value of x, for which $\phi(x)$ is defined, $f(x, \phi(x))$ vanishes identically, *i.e.*, $y = \phi(x)$ is a root of the functional equation $f(x, y) = 0$, then $y = \phi(x)$ is an *implicit function* defined by the functional equation $f(x, y) = 0$.

It is only in elementary cases, such as those given above, that it may be possible to express y as a function of x (*i.e.*, determine the implicit function). For more complicated functional equations no such determination of implicit function is possible. The difficulty of actual determination of an analytical expression does not rule out the possibility of the *existence* of the implicit function or functions, defined by a functional equation; the actual determination may demand new processes or may be, from a practical standpoint, too laborious. We now consider an Existence theorem, known as *Implicit function theorem*, that specifies conditions which guarantee that a functional equation does define an implicit function even though actual determination may not be possible. For many purposes, however, it is the fact that an equation does define a function, rather than an expression for the implicit function thus defined, that is of real importance; hence the value of Existence theorem.

12.1.1 Existence Theorem (*Case of Two Variables*)

Let $f(x, y)$ be a function of two variables x and y, and (a, b) be a point of its domain of definition such that

(*i*) $f(a, b) = 0$,

(*ii*) *the partial derivatives f_x and f_y exist, and are continuous in a certain neighbourhood of (a, b), and*

(*iii*) $f_y(a, b) \neq 0$, *then there exists a rectangle* $(a - h, a + h; b - k, b + k)$ *about* (a, b) *such that for every value of x in the interval* $[a - h, a + h]$, *the equation* $f(x, y) = 0$ *determines one and only one value* $y = \phi(x)$, *lying in the interval* $[b - k, b + k]$, *with the following properties*:

(1) $b = \phi(a)$,

(2) $f[x, \phi(x)] = 0$, *for every x in* $[a - h, a + h]$, *and*

(3) $\phi(x)$ *is derivable, and both* $\phi(x)$ *and* $\phi'(x)$ *are continuous in* $[a - h, a + h]$.

Existence. Let f_x, f_y be continuous in a neighbourhood

$$R_1 : (a - h_1, a + h_1; b - k_1, b + k_1), \text{ of } (a, b)$$

Since f_x, f_y exist and are continuous in R_1, therefore f is differentiable and hence continuous in R_1.

Again, since f_y is continuous, and $f_y(a, b) \neq 0$, there exists a rectangle

$$R_2 : (a - h_2, a + h_2; b - k_2, b + k_2),\ h_2 < h_1,\ k_2 < k_1$$

(R_2 contained in R_1) such that for every point of this rectangle, $f_y \neq 0$.

Since $f = 0$ and $f_y \neq 0$ (it is therefore either positive or negative) at the point (a, b), a positive number $k < k_2$ can be found such that

$$f(a, b - k), f(a, b + k)$$

are of opposite signs, for, f is either an increasing or a decreasing function of y, when $y = b$.

Again, since f is continuous, a positive number $h < h_2$ can be found such that for all x in $[a - h, a + h]$,

$$f(x, b - k), f(x, b + k),$$

respectively, may be as near as we please to $f(a, b - k), f(a, b + k)$ and therefore have opposite signs.

Thus, for all x in $[a - h, a + h]$, f is a continuous function of y and changes sign as y changes from $b - k$ to $b + k$. Therefore it vanishes for some y in $[b - k, b + k]$.

Thus, for each x in $[a - h, a + h]$, there is a y in $[b - k, b + k]$ for which $f(x, y) = 0$; this y is a function of x, say $\phi(x)$ such that properties (1) and (2) are true.

Uniqueness. We, now, show that $y = \phi(x)$ is a unique solution of $f(x, y) = 0$ in $R_3 : (a - h, a + h; b - k, b + k)$; that is , $f(x, y)$ cannot be zero for more than one value of y in $[b - k, b + k]$.

Let, if possible, there be two such values y_1, y_2 in $[b - k, b + k]$ so that $f(x, y_1) = 0, f(x, y_2) = 0$. Also $f(x, y)$ considered as a function of a single variable y is derivable in $[b - k, b + k]$, so that by Rolle's theorem, $f_y = 0$ for a value of y between y_1 and y_2, which contradicts the fact that $f_y \neq 0$ in $R_2 \supset R_3$. Hence our supposition is wrong and there cannot be more than one such y.

Derivability. Let (x, y), $(x + \delta x, y + \delta y)$ be two points in $R_3 : (a - h, a + h; b - k, b + k)$ such that

$$y = \phi(x),\ y + \delta y = \phi(x + \delta x)$$

and

$$f(x, y) = 0,\ f(x + \delta x, y + \delta y) = 0$$

Since f is differentiable in R_1 and consequently in R_3 (contained in R_1),

$\therefore$
$$0 = f(x + \delta x, y + \delta y) - f(x, y)$$
$$= \delta x f_x + \delta y f_y + \delta x \psi_1 + \delta y \psi_2$$

where ψ_1, ψ_2 are function of δx and δy, and tend to 0 as

$$(\delta x, \delta y) \to (0, 0)$$

or

$$\frac{\delta y}{\delta x} = -\frac{f_x}{f_y} - \frac{\psi_1}{f_y} - \frac{\delta y}{\delta x}\frac{\psi_2}{f_y} \qquad (f_y \neq 0 \text{ in } R_3)$$

Proceeding to limits as $(\delta x, \delta y) \to (0, 0)$, we get

$$\phi'(x) = \frac{dy}{dx} = -\frac{f_x}{f_y}$$

Thus $\phi(x)$ is derivable and hence continuous in R_3. Also $\phi'(x)$, being a quotient of two continuous functions, is itself continuous in R_3.

Note. It should be clearly understood that the theorem is essentially of a *local character*. That is, the implicit function $y = \phi(x)$ is a unique solution of $f(x, y) = 0$ in a certain neighbourhood $(a - h, a + h; b - k, b + k)$ of (a, b).

It may have a different solution $y = \psi(x)$ if a different neighbourhood of (a, b) is considered.

Example 12.1. Let $f(x, y) = x^2 + y^2 - 1$, and a point (0, 1) so that

$$f(0, 1) = 0 \text{ and } f_y(0, 1) \neq 0$$

Of the two possible solutions $y = \pm\sqrt{1 - x^2}$.

(*i*) $y = +\sqrt{1 - x^2}$ is the implicit function in a *nbd* of (0, 1), where $|x| < 1$, $y > 0$.

(*ii*) $y = -\sqrt{1 - x^2}$ is the implicit function in a *nbd* of (0, –1), where $|x| < 1$, $y < 0$.

12.1.2 General Case

Let $f(x_1, x_2, ..., x_n, y)$ *be a function of* $(n + 1)$ *variables,* $x_1, x_2, ..., x_n, y$ *and* $(a_1, a_2, ..., a_n, b)$ *be a point of its domain of definition such that*

(*i*) $f(a_1, a_2, ..., a_n, b) = 0$

(*ii*) *the partial derivatives with respect to all the* $(n + 1)$ *variables exist and are continuous in a certain neighbourhood of* $(a_1, a_2, ..., a_n, b)$, *and*

(*iii*) $f_y(a_1, a_2, ..., a_n, b) \neq 0$,

then there exists a neighbourhood

$$(a_1 - h_1, a_1 + h_1; a_2 - h_2, a_2 + h_2; ...; a_n - h_n, a_n + h_n; b - k, b + k)$$

of $(a_1, a_2, ..., a_n, b)$ *such that for every point* $(x_1, x_2, ..., x_n)$ *of the neighbourhood*

$$R : (a_1 - h_1, a_1 + h_1; a_2 - h_2, a_2 + h_2; ...; a_n - h_n, a_n + h_n)$$

the equation $f(x_1, x_2, ..., x_n, y) = 0$ *determines one and only one value* $y = \phi(x_1, x_2, ..., x_n)$ *lying in* $[b - k, b + k]$ *with the following properties*:

(1) $b = \phi(a_1, a_2, ..., a_n)$.

(2) $f(x_1, x_2, ..., x_n, \phi) = 0$ *for every point* $(x_1, x_2, ..., x_n)$ *in* R,

(3) ϕ *is continuous and possesses continuous partial derivatives of the first order with respect to* $x_1, x_2, ..., x_n$ *in* R.

The proof is essentially a repetition of that given for the preceding theorem and offers no fresh difficulties.

EXERCISE

Examine the following equations for the existence of unique implicit function near the points indicated and verify by direct calculation. Also find the first derivatives of the solutions whenever these exist.

1. $y^2 + 2x^2y + x^5 = 0, (1, -1)$
2. $y^2 + yx^3 + x^2 = 0, (0, 0)$
3. $y^3 + x^3 - 3xy + y = 0\ (0, 0)$
4. $2xy - \log xy = 2, (1, 1)$
5. $y^4 + y^2x^2 - 2x^5 = 0, (1, 1)$
6. $y^2 - yx^2 - 2x^5 = 0, (0, 0)$
7. $y^3\cos x + y^2 \sin^2 x - 7 = 0, (\pi/3, 2)$
8. $xy \sin x + \cos y = 0, (0, \pi/2)$

12.1.3 Derivative of Implicit Functions

When the equation $f(x, y) = 0$ defines y as a function of x that has a derivative dy/dx, that derivative may be obtained simply by differentiating the equation with respect to x, on the understanding that y is a function $\phi(x)$ of x. Thus

$$f_x + f_y \frac{dy}{dx} = 0 \qquad ...(1)$$

If the higher partial derivatives of $f(x, y)$ are continuous, we obtain the higher derivatives of y or $\phi(x)$ by successive differentiation of (1), provided always that f_y is not zero; thus

$$f_{xx} + f_{xy} \frac{dy}{dx} + \left(f_{xy} + f_{yy} \frac{dy}{dx} \right) \frac{dy}{dx} + f_y \frac{d^2y}{dx^2} = 0$$

or

$$f_{xx} + 2f_{xy} \frac{dy}{dx} + f_{yy} \left(\frac{dy}{dx} \right)^2 + f_y \frac{d^2y}{dx^2} = 0 \qquad ...(2)$$

Provided f_y is not zero, this equation determines the second derivative, and in a similar way the third and higher derivatives may be found.

Note. If $f(x, y) = 0$, and y is a function of x, we have from (1)

$$\frac{dy}{dx} = \frac{-p}{q}$$

and also from (2)

$$\frac{d^2y}{dx^2} = -\frac{r + 2s(-p/q) + t(-p/q)^2}{q} = \frac{-(rq^2 - 2spq + tp^2)}{q^3}$$

12.2 JACOBIANS

For further development of the subject, acquaintance with the notion of Jacobians is necessary. We shall now define a Jacobian and also prove some of its important properties.

If $u_1, u_2, ..., u_n$ be n differentiable functions of n variables $x_1, x_2, ..., x_n$, then the determinant

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots \quad \vdots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called the *Jacobian* or the *Functional Determinant* of the functions $u_1, u_2, ..., u_n$ with respect to $x_1, x_2, ..., x_n$ and is denoted by

$$\frac{\partial (u_1, u_2, ..., u_n)}{\partial (x_1, x_2, ..., x_n)} \text{ or } J\left(\frac{u_1, u_2, ..., u_n}{x_1, x_2, ..., x_n}\right).$$

12.2.1 Some Properties

Jacobians have the remarkable property of behaving like the derivatives of functions of one variable. A few of the important relations are given here and the proofs depend upon the algebra of determinants.

For $n = 1$, the determinant is simply $\frac{\partial y_1}{\partial x_1}$ or $\frac{dy_1}{dx_1}$, the derivative of y_1 with respect to x_1; the first of the notations for a Jacobian is suggested by a certain analogy between the properties of the Jacobian and the derivative.

Theorem 12.1. *If $u_1, u_2, ..., u_n$ are functions of $y_1, y_2, ..., y_n$ and $y_1, y_2, ..., y_n$ are themselves functions of $x_1, x_2, ..., x_n$, then*

$$\frac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)} = \frac{\partial(u_1, u_2, ..., u_n)}{\partial(y_1, y_2, ..., y_n)} \frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ..., x_n)} \quad ...(1)$$

For $n = 1$, the theorem reduces to the usual notation

$$\frac{du_1}{dx_1} = \frac{du_1}{dy_1} \frac{dy_1}{dx_1}$$

The proof of the theorem depends on the "row by column" rule of multiplication of determinants combined with the rule for the derivative of a function.

Thus, for determinants on the right hand side of (1), rth row of the first is $\frac{\partial u_r}{\partial y_1}, \frac{\partial u_r}{\partial y_2}, ..., \frac{\partial u_r}{\partial y_n}$, sth column of the second is $\frac{\partial y_1}{\partial x_s}, \frac{\partial y_2}{\partial x_s}, ..., \frac{\partial y_n}{\partial x_s}$, so that the element in the rth row and the sth column of the product is

$$\frac{\partial u_r}{\partial y_1}\frac{\partial y_1}{\partial x_s} + \frac{\partial u_r}{\partial y_2}\frac{\partial y_2}{\partial x_s} + ... + \frac{\partial u_r}{\partial y_n}\frac{\partial y_n}{\partial x_s}$$

and this is equal to $\frac{\partial u_r}{\partial x_s}$, which is the element in the rth row and the sth column of the Jacobian on the left hand side. Hence, the theorem.

Corollary. If $x_r = u_r$, $r = 1, 2, ..., n$ and assuming the existence of inverse functions $x_1, x_2, ..., x_n$ (that is, assuming that the equations which define $y_1, y_2, ..., y_n$ as functions of $x_1, x_2, ..., x_n$ determine $x_1, x_2, ..., x_n$ as functions of $y_1, y_2, ..., y_n$) we find

$$\frac{\partial(x_1, x_2, ..., x_n)}{\partial(y_1, y_2, ..., y_n)} \cdot \frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ..., x_n)} = \frac{\partial(x_1, x_2, ..., x_n)}{\partial(x_1, x_2, ..., x_n)} = 1, \qquad ...(2)$$

since $\frac{\partial x_i}{\partial x_j} = 0$, for $i \neq j$ $= 1$, for $i = j$

Theorem 12.2. *If $y_1, y_2, ..., y_n$ are determined as functions of $x_1, x_2, ..., x_n$ by the equations*

$$\phi_r(x_1, x_2, ..., x_n, y_1, y_2, ..., y_n) = 0, \; r = 1, 2, ..., n$$

then

$$\frac{\partial(\phi_1, \phi_2, ..., \phi_n)}{\partial(x_1, x_2, ..., x_n)} = (-1)^n \frac{\partial(\phi_1, \phi_2, ..., \phi_n)}{\partial(y_1, y_2, ..., y_n)} \frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ..., x_n)} \qquad ...(3)$$

[Theorem 12.1 is a particular form of this theorem.]

Differentiating the equations $\phi_r = 0$, with respect to x_s, we get

$$\frac{\partial \phi_r}{\partial x_s} + \frac{\partial \phi_r}{\partial y_1}\frac{\partial y_1}{\partial x_s} + \frac{\partial \phi_r}{\partial y_2}\frac{\partial y_2}{\partial x_s} + ... + \frac{\partial \phi_r}{\partial y_n}\frac{\partial y_n}{\partial x_s} = 0$$

or

$$\frac{\partial \phi_r}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_s} + \frac{\partial \phi_r}{\partial y_2}\frac{\partial y_2}{\partial x_s} + ... + \frac{\partial \phi_r}{\partial y_n}\frac{\partial y_n}{\partial x_s} = -\frac{\partial \phi_r}{\partial x_s}$$

so that the element in the rth row and the sth column of the determinant which is the product of the two determinants on the right of (3), is $-\frac{\partial \phi_r}{\partial x_s}$, from which the result follows.

Theorem 12.3. *(i) If $y_{m+1}, y_{m+2}, ..., y_n$ are constant with respect to $x_1, x_2, ..., x_m$, or (ii) if $y_1, y_2, ..., y_m$ are constant with respect to $x_{m+1}, x_{m+2}, ..., x_n$, then*

$$\frac{\partial(y_1, y_2, ..., y_m, ..., y_n)}{\partial(x_1, x_2, ..., x_m, ..., x_n)} = \frac{\partial(y_1, y_2, ..., y_m)}{\partial(x_1, x_2, ..., x_m)} \cdot \frac{\partial(y_{m+1}, y_{m+2}, ..., y_n)}{\partial(x_{m+1}, x_{m+2}, ..., x_n)} \qquad ...(4)$$

(*i*) $\dfrac{\partial y_r}{\partial x_s} = 0$, when $r = m + 1, m + 2, ..., n$; $s = 1, 2, ..., m$.

$$\therefore \quad \frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ..., x_n)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots \frac{\partial y_1}{\partial x_m} & \frac{\partial y_1}{\partial x_{m+1}} & \cdots \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots \frac{\partial y_2}{\partial x_m} & \frac{\partial y_2}{\partial x_{m+1}} & \cdots \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \vdots\ \vdots & \vdots & \vdots\ \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots \frac{\partial y_m}{\partial x_m} & \frac{\partial y_m}{\partial x_{m+1}} & \cdots \frac{\partial y_m}{\partial x_n} \\ \frac{\partial y_{m+1}}{\partial x_1} & \frac{\partial y_{m+1}}{\partial x_2} & \cdots \frac{\partial y_{m+1}}{\partial x_m} & \frac{\partial y_{m+1}}{\partial x_{m+1}} & \cdots \frac{\partial y_{m+1}}{\partial x_n} \\ \vdots & \vdots & \vdots\ \vdots & \vdots & \vdots\ \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots \frac{\partial y_n}{\partial x_m} & \frac{\partial y_n}{\partial x_{m+1}} & \cdots \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots \frac{\partial y_1}{\partial x_m} & \frac{\partial y_1}{\partial x_{m+1}} & \cdots \frac{\partial y_m}{\partial x_n} \\ \vdots & \vdots & \vdots\ \vdots & \vdots & \vdots\ \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots \frac{\partial y_m}{\partial x_m} & \frac{\partial y_m}{\partial x_{m+1}} & \cdots \frac{\partial y_m}{\partial x_n} \\ 0 & 0 & \cdots 0 & \frac{\partial y_{m+1}}{\partial x_{m+1}} & \cdots \frac{\partial y_{m+1}}{\partial x_n} \\ 0 & 0 & \cdots 0 & \frac{\partial y_{m+2}}{\partial x_{m+1}} & \cdots \frac{\partial y_{m+2}}{\partial x_n} \\ \vdots & \vdots & \vdots\ \vdots & \vdots & \vdots\ \vdots \\ 0 & 0 & \cdots 0 & \frac{\partial y_n}{\partial x_{m+1}} & \cdots \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

$$= \frac{\partial(y_1, y_2, ..., y_m)}{\partial(x_1, x_2, ..., x_n)} \cdot \frac{\partial(y_{m+1}, y_{m+2}, ..., y_n)}{\partial(x_{m+1}, x_{m+2}, ..., x_n)}$$

(*ii*) may also be proved similarly.

Corollary. In particular,

$$\frac{\partial(y_1, ..., y_m, x_{m+1}, ..., x_n)}{\partial(x_1, ..., x_m, x_{m+1}, ..., x_n)} = \frac{\partial(y_1, ..., y_m)}{\partial(x_1, ..., x_m)} \quad ...(5)$$

Theorem 12.4. *If u, v are functions of* ξ, η, ζ, *and the variables* ξ, η, ζ, *are themselves functions of the independent variables x and y, then*

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(\xi, \eta)} \cdot \frac{\partial(\xi, \eta)}{\partial(x, y)} + \frac{\partial(u, v)}{\partial(\eta, \xi)} \cdot \frac{\partial(\eta, \xi)}{\partial(x, y)} + \frac{\partial(u, v)}{\partial(\zeta, \xi)} \cdot \frac{\partial(\zeta, \xi)}{\partial(x, y)} \quad ...(6)$$

We have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial x} \quad ...(7)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial y} \quad ...(8)$$

and if we substitute these values in the Jacobian $\frac{\partial(u, v)}{\partial(x, y)}$, we get

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial u}{\partial \xi}\frac{\partial(\xi, v)}{\partial(x, y)} + \frac{\partial u}{\partial \eta}\frac{\partial(\eta, v)}{\partial(x, y)} + \frac{\partial u}{\partial \zeta}\frac{\partial(\zeta, v)}{\partial(x, y)} \quad ...(9)$$

which is a linear expression of the Jacobians of $(\xi, v), (\eta, v)$ and (ζ, v) with respect to x and y.

Now in each Jacobian on the right of (9), substitute the expressions for $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ which are similar to equations (7) and (8). Each of these Jacobians will be given as a linear expression of the Jacobians of $(\xi, \eta), (\eta, \zeta)$ and (ζ, ξ) since those of $(\xi, \xi), (\eta, \eta)$ and (ζ, ζ) have two identical parallel lines and so vanish. Thus we see that the terms which involve the Jacobian of (ξ, η) are

$$\frac{\partial u}{\partial \xi}\frac{\partial v}{\partial \eta}\frac{\partial(\xi, \eta)}{\partial(x, y)} + \frac{\partial u}{\partial \eta}\frac{\partial v}{\partial \xi}\frac{\partial(\eta, \xi)}{\partial(x, y)}$$

which is equal to $\frac{\partial(u, v)}{\partial(\xi, \eta)}\frac{\partial(\xi, \eta)}{\partial(x, y)}$, the first terms on the right of equation (6).

Similarly we obtain the remaining two terms and the formula is established.

Example 12.2. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, then show that

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin\theta.$$

Solution.

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2 \sin\theta \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\theta \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

Adding $(\cos\phi)\ R_1$ to $(\sin\phi)\ R_2$,

$$= \frac{r^2 \sin\theta}{\sin\phi} \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta & \cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \end{vmatrix} = r^2 \sin\theta.$$

Example 12.3. If $y_1 + y_2 + ... + y_n = x_1$, $y_2 + y_3 + ... + y_n = x_1x_2$,..., $y_r + y_{r+1} + ... + y_n = x_1x_2 ... x_r$, ..., $y_n = x_1x_2 ... x_n$, then show that

$$\frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ..., x_n)} = x_1^{n-1} x_2^{n-2} ... x_{n-2}^2 x_{n-1}.$$

Solution. Solving for y_1, y_2, ..., y_n, we get

$$\begin{aligned} y_1 &= x_1 - x_1x_2 = x_1\ (1 - x_2) \\ y_2 &= x_1x_2 - x_1x_2x_3 = x_1\ x_2\ (1 - x_3) \\ &\vdots \\ y_{n-1} &= x_1x_2\ ...\ x_{n-1}\ (1 - x_n) \\ y_n &= x_1x_2\ ...\ x_n. \end{aligned}$$

$$\therefore \quad \frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ..., x_n)}$$

$$= \begin{vmatrix} 1-x_2 & -x_1 & 0 & ... & 0 \\ x_2(1-x_3) & x_1(1-x_3) & -x_1x_2 & ... & 0 \\ x_2x_3(1-x_4) & x_1x_3(1-x_4) & x_1x_2(1-x_4) & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_2x_3...x_{n-1}(1-x_n) & x_1x_3...x_{n-1}(1-x_n) & x_1x_2x_4...x_{n-1}(1-x_n) & ... & x_1x_2...x_{n-1} \\ x_2x_3...x_n & x_1x_2x_4...x_n & x_1x_3...x_n & ... & x_1x_2...x_{n-1} \end{vmatrix}$$

Adding, R_n to R_{n-1}, then R_{n-1} to R_{n-2}, ..., then R_2 to R_1 and expanding by last column

$$\begin{aligned} &= (x_1x_2...x_{n-1})\ (x_1x_2...x_{n-2})...(x_1x_2)(x_1) \\ &= x_1^{n-1} x_2^{n-2} ... x_{n-2}^2 x_{n-1}. \end{aligned}$$

Example 12.4. The roots of the equation in λ

$$(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0,$$

are u, v, w. Prove that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = -2\frac{(y - z)(z - x)(x - y)}{(v - w)(w - u)(u - v)}.$$

Solution. Here u, v, w are roots of the equation

$$\lambda^3 - (x + y + z)\lambda^2 + (x^2 + y^2 + z^2)\lambda - \frac{1}{3}(x^3 + y^3 + z^3) = 0$$

Let
$$x + y + z = \xi,$$
$$x^2 + y^2 + z^2 = \eta,$$
$$\frac{1}{3}(x^3 + y^3 + z^3) = \zeta \qquad \ldots(1)$$

and

$$u + v + w = \xi,$$
$$vw + wu + uv = \eta,$$
$$uvw = \zeta \qquad \ldots(2)$$

Hence, from (1),

$$\frac{\partial(\xi, \eta, \zeta)}{\partial(x, y, z)} = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ x^2 & y^2 & z^2 \end{vmatrix}$$
$$= 2(y - z)\ (z - x)\ (x - y) \qquad \ldots(3)$$

and from (2),

$$\frac{\partial(\xi, \eta, \zeta)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 1 & 1 \\ v + w & w + u & u + v \\ vw & wu & uv \end{vmatrix}$$
$$= -(v - w)\ (w - u)\ (u - v) \qquad \ldots(4)$$

Hence, from (3) and (4) and using Theorem 12.1, we get

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{\partial(u, v, w)}{\partial(\xi, \eta, \zeta)} \cdot \frac{\partial(\xi, \eta, \zeta)}{\partial(x, y, z)} = -2\frac{(y - z)(z - x)(x - y)}{(v - w)(w - u)(u - v)}.$$

Example 12.5. If $y_r = \dfrac{u_r}{u}$, $r = 1, 2, \ldots, n$, and if u and u_r are functions of the n independent variables $x_1, x_2, \ldots, x_n$, prove that

$$\frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} = \frac{1}{u^{n+1}} \begin{vmatrix} u & \dfrac{\partial u}{\partial x_1} & \dfrac{\partial u}{\partial x_2} & \ldots & \dfrac{\partial u}{\partial x_n} \\ u_1 & \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \ldots & \dfrac{\partial u_1}{\partial x_n} \\ u_2 & \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \ldots & \dfrac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_n & \dfrac{\partial u_n}{\partial x_1} & \dfrac{\partial u_n}{\partial x_2} & \ldots & \dfrac{\partial u_n}{\partial x_n} \end{vmatrix}$$

Solution. Now

$$\frac{\partial y_r}{\partial x_s} = \frac{1}{u}\frac{\partial u_r}{\partial x_s} - \frac{u_r}{u^2}\frac{\partial u}{\partial x_s}$$

$$\therefore \qquad \frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ..., x_n)} = \begin{vmatrix} \frac{1}{u}\frac{\partial u_1}{\partial x_1} - \frac{u_1}{u^2}\frac{\partial u}{\partial x_1} & \cdots & \frac{1}{u}\frac{\partial u_1}{\partial x_n} - \frac{u_1}{u^2}\frac{\partial u}{\partial x_n} \\ \frac{1}{u}\frac{\partial u_2}{\partial x_1} - \frac{u_2}{u^2}\frac{\partial u}{\partial x_1} & \cdots & \frac{1}{u}\frac{\partial u_2}{\partial x_n} - \frac{u_2}{u^2}\frac{\partial u}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{1}{u}\frac{\partial u_n}{\partial x_1} - \frac{u_n}{u^2}\frac{\partial u}{\partial x_1} & \cdots & \frac{1}{u}\frac{\partial u_n}{\partial x_n} - \frac{u_n}{u^2}\frac{\partial u}{\partial x_n} \end{vmatrix}$$

Taking out $\frac{1}{u}$ from each column and bordering the determinant, we get

$$\frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ..., x_n)} = \frac{1}{u^n}\begin{vmatrix} 1 & 0 & \cdots & 0 \\ u_1 & \frac{\partial u_1}{\partial x_1} - \frac{u_1}{u}\frac{\partial u}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_n} - \frac{u_1}{u}\frac{\partial u}{\partial x_n} \\ u_2 & \frac{\partial u_2}{\partial x_1} - \frac{u_2}{u}\frac{\partial u}{\partial x_1} & \cdots & \frac{\partial u_2}{\partial x_n} - \frac{u_2}{u}\frac{\partial u}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ u_n & \frac{\partial u_n}{\partial x_1} - \frac{u_n}{u}\frac{\partial u}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_n} - \frac{u_n}{u}\frac{\partial u}{\partial x_n} \end{vmatrix}$$

Operating $\quad C_2 + \frac{1}{u}\frac{\partial u}{\partial x_1}C_1, C_3 + \frac{1}{u}\frac{\partial u}{\partial x_2}C_1, ..., C_{n+1} + \frac{1}{u}\frac{\partial u}{\partial x_n}C_1$

$$= \frac{1}{u^n}\begin{vmatrix} 1 & \frac{1}{u}\frac{\partial u}{\partial x_1} & \cdots & \frac{1}{u}\frac{\partial u}{\partial x_n} \\ u_1 & \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_n} \\ u_2 & \frac{\partial u_2}{\partial x_1} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ u_n & \frac{\partial u_n}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

$$= \frac{1}{u^{n+1}} \begin{vmatrix} u & \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} & \dots & \frac{\partial u}{\partial x_n} \\ u_1 & \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \dots & \frac{\partial u_1}{\partial x_n} \\ u_2 & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \dots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_n & \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \dots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

EXERCISE

1. If $u = \cos x$, $v = \sin x \cos y$, $w = \sin x \sin y \cos z$, then show that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^2 \sin^3 x \sin^2 y \sin z.$$

2. If $u = a \cosh x \cos y$, $v = a \sinh x \sin y$, then show that

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{1}{2}a^2(\cosh 2x - \cos 2y).$$

3. If $x + y + z = u$, $y + z = uv$, $z = uvw$, then show that

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = u^2 v.$$

4. If α, β, γ are the roots of the equation in t, such that

$$\frac{u}{a+t} + \frac{v}{b+t} + \frac{w}{c+t} = 1,$$

then prove that

$$\frac{\partial(u, v, w)}{\partial(\alpha, \beta, \gamma)} = -\frac{(\beta - \gamma)(\gamma - \alpha)(\alpha - \beta)}{(b - c)(c - a)(a - b)}.$$

5. If $u = x/(1 - r^2)^{1/2}$, $v = y/(1 - r^2)^{1/2}$, $w = z/(1 - r^2)^{1/2}$ where $r^2 = x^2 + y^2 + z^2$, then show that

$$J\begin{pmatrix} u, v, w \\ x, y, z \end{pmatrix} = \frac{1}{(1 - r^2)^{5/2}}.$$

12.3 STATIONARY VALUES UNDER SUBSIDIARY CONDITIONS

To find the stationary values of the function

$$f(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m) \qquad \dots(1)$$

of $(n + m)$ *variables which are connected by m differentiable equations*

$$\phi_r(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m) = 0,\ r = 1, 2, \dots, m \qquad \dots(2)$$

If the m variables $u_1, u_2, ..., u_m$ are determinate as functions of $x_1, x_2, ..., x_n$ from the system of m equations (2), then f can be regarded as a function of n independent variables $x_1, x_2, ..., x_n$.

At a stationary point of f, $df = 0$.

Hence, at a stationary point (by § 11.1 Ch. 11).

$$0 = df = f_{x_1}dx_1 + f_{x_2}dx_2 + ... + f_{x_n}dx_n + f_{u_1}du_1 + ... + f_{u_m}du_m \qquad ...(3)$$

Again, differentiating the equation (2), we get

$$\left.\begin{aligned} &\frac{\partial \phi_1}{\partial x_1}dx_1 + ... + \frac{\partial \phi_1}{\partial x_n}dx_n + \frac{\partial \phi_1}{\partial u_1}du_1 + ... + \frac{\partial \phi_1}{\partial u_m}du_m = 0 \\ &\frac{\partial \phi_2}{\partial x_1}dx_1 + ... + \frac{\partial \phi_2}{\partial x_n}dx_n + \frac{\partial \phi_2}{\partial u_1}du_1 + ... + \frac{\partial \phi_2}{\partial u_m}du_m = 0 \\ &\quad\vdots \\ &\frac{\partial \phi_m}{\partial x_1}dx_1 + ... + \frac{\partial \phi_m}{\partial x_n}dx_n + \frac{\partial \phi_m}{\partial u_1}du_1 + ... + \frac{\partial \phi_m}{\partial u_m}du_m = 0 \end{aligned}\right\} \qquad ...(4)$$

From these m equations (4), the differentials $du_1, du_2, ..., du_m$ of the m dependent variables may be found in terms of the n differentials $dx_1, dx_2, ..., dx_n$ and substituted in (3). This way df has been expressed in terms of the differentials of the independent variables, and since the differentials of the independent variables are arbitrary and $df = 0$, the coefficients of each of these n differentials may be equated to zero. These n equations together with the m equations (2) constitute a system of $(n + m)$ equations to determine the $(n + m)$ coordinates of the stationary points of f.

Example 12.6. $F(x, y, z)$ is a function subject to the constraint condition $G(x, y, z) = 0$. Show that at a stationary point

$$F_xG_y - F_yG_x = 0.$$

Solution. We may consider z as a function of the independent variables x, y.

As a stationary point, $dF = 0$

$$\therefore \qquad 0 = dF = F_xdx + F_ydy + F_zdz \qquad ...(i)$$

Differentiating the relation $G(x, y, z) = 0$, we get

$$G_xdx + G_ydy + G_zdz = 0 \qquad ...(ii)$$

Putting the value of dz from (2) into (1), or what is same thing, eliminating dz from (1) and (2), we get

$$(F_xG_z - G_xF_z)\,dx + (F_yG_z - G_yF_z)\,dy = 0 \qquad ...(iii)$$

Since dx, dy (being differentials of independent variables) are arbitrary, therefore

$$F_xG_z - G_xF_z = 0$$

$$F_yG_z - G_yF_z = 0$$

which give

$$F_xG_y - F_yG_x = 0$$

EXERCISE

1. Find the stationary points of the function xy^2z^2 subject to the conditions

$$x + y + z = 6,\ x > 0,\ y > 0,\ z > 0.$$

2. Find the stationary points of $x^2 + y^2$ subject to the condition

$$3x^2 + 4xy + 6y^2 = 140.$$

3. Find the stationary points of the function $x^2y^2z^2$ subject to the condition

$$x^2 + y^2 + z^2 = a^2.$$

4. Find the stationary points of the function xyz, when x, y, z are connected by the equation

$$x^2/9 + y^2/16 + z^2/36 = 1.$$

12.3.1 Lagrange's Undetermined Multipliers

In this section we shall discuss the determination of stationary points from a modified point of view. The process consists in the introduction of undetermined multipliers, a method due to Lagrange.

Before we discuss the method proper, let us notice that in the above section (§ 3) the differentials $du_1, du_2, ..., du_m$ of the m dependent variables were found from equations (4) and substituted in (3), so as to express df in terms of the differentials of the independent variables. All this amounts to elimination of the differentials of the dependent variables from (3) and (4). This process of elimination is effected due to Lagrange by the introduction of multipliers. The process is as follows:

Multiply each of the equations (4) by $\lambda_1, \lambda_2, ..., \lambda_m$, which are to be specified later, and add to (3). Now the m multipliers are so chosen that the coefficients of the m differentials of dependent variables all vanish. This gives m equations to determine λ's. Thus is equation (3) there remain only the differentials of n independent variables, the coefficient of each one of them may be equated to zero. The process thus gives $n + m$ equations which along with m constraint conditions form $n + 2m$ equations, to determine the m multipliers $\lambda_1, \lambda_2, ..., \lambda_m$, and $m + n$ variables $x_1, x_2, ..., x_n, u_1, u_2, ..., u_m$, for which the function has a stationary value.

12.3.2 Lagrange's Method of Multipliers

To find the stationary points of the function

$$f(x_1, x_2, ..., x_n, u_1, u_2, ..., u_m), \qquad ...(1)$$

of $n + m$ variables which are connected by the equations

$$\phi_r(x_1, x_2, ..., x_n, u_1, u_2, ..., u_m) = 0,\ r = 1, 2, ..., m \qquad ...(2)$$

As in § 3, if the m variables $u_1, u_2, ..., u_m$ from equations (2) are expressed in terms of variables $x_1, x_2, ..., x_n$ the function f may be considered as a function of n independent variables

$$x_1, x_2, ..., x_n.$$

For stationary values, $df = 0$

$$\therefore \qquad 0 = df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + ... + \frac{\partial f}{\partial x_n}dx_n + \frac{\partial f}{\partial u_1}du_1 + ... + \frac{\partial f}{\partial u_m}du_m \qquad ...(3)$$

Differentiating equations (2), we get

$$\left.\begin{aligned}
&\frac{\partial \phi_1}{\partial x_1} dx_1 + ... + \frac{\partial \phi_1}{\partial x_n} dx_n + \frac{\partial \phi_1}{\partial u_1} du_1 + ... + \frac{\partial \phi_1}{\partial u_m} du_m = 0 \\
&\frac{\partial \phi_2}{\partial x_1} dx_1 + ... + \frac{\partial \phi_2}{\partial x_n} dx_n + \frac{\partial \phi_2}{\partial u_1} du_1 + ... + \frac{\partial \phi_2}{\partial u_m} du_m = 0 \\
&\vdots \\
&\frac{\partial \phi_m}{\partial x_1} dx_1 + ... + \frac{\partial \phi_m}{\partial x_n} dx_n + \frac{\partial \phi_m}{\partial u_1} du_1 + ... + \frac{\partial \phi_m}{\partial u_m} du_m = 0
\end{aligned}\right\} \quad ...(4)$$

Multiplying the equations (4) by $\lambda_1, \lambda_2, ..., \lambda_m$ respectively and adding to the equation (3), we get

$$0 = df = \left(\frac{\partial f}{\partial x_1} + \Sigma \lambda_r \frac{\partial \phi_r}{\partial x_1} \right) dx_1 + ... + \left(\frac{\partial f}{\partial x_n} + \Sigma \lambda_r \frac{\partial \phi_r}{\partial x_n} \right) dx_n$$

$$+ \left(\frac{\partial f}{\partial u_1} + \Sigma \lambda_r \frac{\partial \phi_r}{\partial u_1} \right) du_1 + ... + \left(\frac{\partial f}{\partial u_m} + \Sigma \lambda_r \frac{\partial \phi_r}{\partial u_m} \right) du_m \quad ...(5)$$

Let the m multipliers $\lambda_1, \lambda_2, ..., \lambda_m$ be so chosen that the coefficients of the m differentials $du_1, du_2, ..., du_m$ all vanish, *i.e.*,

$$\frac{\partial f}{\partial u_1} + \Sigma \lambda_r \frac{\partial \phi_r}{\partial u_1} = 0, ..., \frac{\partial f}{\partial u_m} + \Sigma \lambda_r \frac{\partial \phi_r}{\partial u_m} = 0 \quad ...(6)$$

Then (5) becomes

$$0 = df = \left(\frac{\partial f}{\partial x_1} + \Sigma \lambda_r \frac{\partial \phi_r}{\partial x_1} \right) dx_1 + ... + \left(\frac{\partial f}{\partial x_n} + \Sigma \lambda_r \frac{\partial \phi_r}{\partial x_n} \right) dx_n$$

so that the differential df is expressed in terms of the differentials of independent variables only. Hence

$$\frac{\partial f}{\partial x_1} + \Sigma \lambda_r \frac{\partial \phi_r}{\partial x_1} = 0, ..., \frac{\partial f}{\partial x_n} + \Sigma \lambda_r \frac{\partial \phi_r}{\partial x_n} = 0 \quad ...(7)$$

Equations (2), (6), (7) form a system of $n + 2m$ equations which may be simultaneously solved to determine the m multipliers $\lambda_1, \lambda_2, ..., \lambda_m$ and the $n + m$ coordinates $x_1, x_2, ..., x_n, u_1, u_2, ..., u_m$ of the stationary points of f.

An Important Rule: For practical purposes, the process of obtaining equations (6) and (7) of the above section, may be put in a precise form as follows:

Define a function

$$F = f + \lambda_1 \phi_1 + \lambda_2 \phi_2 + ... + \lambda_m \phi_m$$

and consider all the variables $x_1, x_2, ..., x_n, u_1, u_2, ..., u_m$ as independent.

At a stationary point of F, $dF = 0$. Therefore

$$0 = dF = \frac{\partial F}{\partial x_1}dx_1 + \frac{\partial F}{\partial x_2}dx_2 + ... + \frac{\partial F}{\partial x_n}dx_n + \frac{\partial F}{\partial u_1}du_1 + ... + \frac{\partial F}{\partial u_m}du_m$$

$$\therefore \qquad \frac{\partial F}{\partial x_1} = 0, ..., \frac{\partial F}{\partial x_n} = 0, \frac{\partial F}{\partial u_1} = 0, ..., \frac{\partial F}{\partial u_m} = 0$$

which are same as equations (7) and (6).

Thus, the stationary points of f may be found by determining the stationary points of the function F, where

$$F = f + \lambda_1\phi_1 + \lambda_2\phi_2 + ... + \lambda_m\phi_m$$

and considering all the variables as independent variables.

A stationary point will be an extreme point of f if d^2F keeps the same sign, and will be a maxima or minima according as d^2F is negative or positive.

Note. It may be easier to deal with d^2F by expressing it in terms of two variables only or by expressing $dx\,dy$, $dy\,dz$, $dz\,dx$ in terms of dx^2, dy^2, dz^2, with the help of constraint conditions. Solved examples will illustrate the procedure.

Example 12.7. Find the shortest distance from the origin to the hyperbola

$$x^2 + 8xy + 7y^2 = 225,\ z = 0.$$

Solution. We have to find the minimum value of $x^2 + y^2$ (the square of the distance from the origin to any point in the xy plane) subject to the constraint

$$x^2 + 8xy + 7y^2 = 225.$$

Consider the function

$$F = x^2 + y^2 + \lambda(x^2 + 8xy + 7y^2 - 225)$$

where x, y are independent variables and λ a constant.

$$dF = (2x + 2x\lambda + 8y\lambda)\,dx + (2y + 8x\lambda + 14y\lambda)\,dy$$

$$\therefore \qquad \left.\begin{aligned}(1+\lambda)x + 4\lambda y = 0\\ 4\lambda x + (1+7\lambda)y = 0\end{aligned}\right\} \ \therefore\ \lambda = 1, -\frac{1}{9}.$$

For $\lambda = 1$, $x = -2y$ and substitution in $x^2 + 8xy + 7y^2 = 225$, gives $y^2 = -45$, for which no real solution exists.

For $\lambda = -\frac{1}{9}$, $y = 2x$ and substitution in $x^2 + 8xy + 7y^2 = 225$, gives $x^2 = 5$, $y^2 = 20$, and so $x^2 + y^2 = 25$.

$$d^2F = 2(1+\lambda)dx^2 + 16\lambda\,dx\,dy + 2(1+7\lambda)dy^2$$

$$= \frac{16}{9}dx^2 - \frac{16}{9}dx\,dy + \frac{4}{9}dy^2, \text{ at } \lambda = -\frac{1}{9}$$

$$= \frac{4}{9}(2dx - dy)^2 > 0,$$

and cannot vanish because $(dx, dy) \neq (0, 0)$.

Hence, the function $x^2 + y^2$ has a minimum value 25.

Note. Here F is a function of two variables and so its maximum or minimum values can be verified by the method of functions of two variables, $AC - B^2 > 0$ also.

Example 12.8. Find the maximum and minimum values of $x^2 + y^2 + z^2$ subject to the conditions $\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$, and $z = x + y$.

Solution. Let us consider a function F of independent variables x, y, z where

$$F = x^2 + y^2 + z^2 + \lambda_1\left(\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} - 1\right) + \lambda_2(x + y - z)$$

$$\therefore \qquad dF = \left(2x + \frac{x}{2}\lambda_1 + \lambda_2\right)dx + \left(2y + \frac{2y}{5}\lambda_1 + \lambda_2\right)dy + \left(2z + \frac{2z}{25}\lambda_1 - \lambda_2\right)dz$$

As x, y, z are independent variables, we get

$$2x + \frac{x}{2}\lambda_1 + \lambda_2 = 0$$

$$2y + \frac{2y}{5}\lambda_1 + \lambda_2 = 0$$

$$2z + \frac{2z}{25}\lambda_1 - \lambda_2 = 0$$

$$\therefore \qquad x = \frac{-2\lambda_2}{\lambda_1 + 4}, \quad y = \frac{-5\lambda_2}{2\lambda_1 + 10}, \quad z = \frac{25\lambda_2}{2\lambda_2 + 50}$$

Substituting in $x + y = z$, we get

$$\frac{2}{\lambda_1 + 4} + \frac{5}{2\lambda_1 + 10} + \frac{25}{2\lambda_1 + 50} = 0, \ \lambda_2 \neq 0 \qquad \ldots(1)$$

for if, $\lambda_2 = 0$, $x = y = z = 0$, but (0, 0, 0) does not satisfy the other condition of constraint.

Hence, from (1), $17\lambda_1^2 + 245\lambda_1 + 750 = 0$, so that $\lambda_1 = -10, -75/17$.

For $\lambda_1 = -10$,

$$x = \frac{1}{3}\lambda_2, \ y = \frac{1}{2}\lambda_2, \ z = \frac{5}{6}\lambda_2$$

Substituting in $\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$, we get

$$\lambda_2^2 = 180/19 \text{ or } \lambda_2 = \pm 6\sqrt{5/19}$$

The corresponding stationary points are

$$(2\sqrt{5/19}, 3\sqrt{5/19}, 5\sqrt{5/19}), (-2\sqrt{5/19}, -3\sqrt{5/19}, -5\sqrt{5/19})$$

The value of $x^2 + y^2 + z^2$ corresponding to these point is 10.

For $\lambda_1 = -75/17$,

$$x = \frac{34}{7}\lambda_2,\ y = -\frac{17}{4}\lambda_2,\ z = \frac{17}{28}\lambda_2$$

which on substitution in $\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$, give

$$\lambda_2 = \pm 140/(17\sqrt{646})$$

The corresponding stationary points are

$$(40/\sqrt{646}, -35/\sqrt{646}, 5/\sqrt{646}),\ (-40/\sqrt{646}, 35/\sqrt{646}, -5/\sqrt{646})$$

The value of $x^2 + y^2 + z^2$ corresponding to these points is 75/17.

Thus, the maximum value is 10 and the minimum 75/17.

Notes 1. We have not theoretically established the existence of maximum or minimum value. We have simply shown that of all the possible values, 10 is the maximum and 75/17 the minimum.

2. Using constraint conditions, $dz = dx + dy;\ \frac{x}{4}dx + \frac{y}{5}dy + \frac{z}{25}dz = 0$, so that dz, dy and consequently d^2F may be expressed in terms of dx (or dx^2) alone. It can, then, be easily verified that 10 is a maximum value and 75/17 the minimum.

Example 12.9. Prove that the volume of the greatest rectangular parallelopiped, that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, is $\frac{8abc}{3\sqrt{3}}$.

Solution. We have to find the greatest value of $8xyz$ subject to the conditions

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,\ x > 0,\ y > 0,\ z > 0 \qquad ...(1)$$

Let us consider a function F of three independent variables x, y, z, where

$$F = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

$$\therefore \qquad dF = \left(8yz + \frac{2x\lambda}{a^2}\right)dx + \left(8zx + \frac{2y\lambda}{b^2}\right)dy + \left(8xy + \frac{2z\lambda}{c^2}\right)dz$$

At stationary points,

$$\left.\begin{aligned} 8yz + \frac{2x\lambda}{a^2} &= 0, \\ 8zx + \frac{2y\lambda}{b^2} &= 0, \\ 8xy + \frac{2z\lambda}{c^2} &= 0 \end{aligned}\right\} \qquad ...(2)$$

Multiplying by x, y, z respectively and adding,

$$24xyz + 2\lambda = 0 \text{ or } \lambda = -12xyz. \qquad \text{[using (1)]}$$

Hence, from (2), $x = a/\sqrt{3}$, $y = b/\sqrt{3}$, $z = c/\sqrt{3}$, and so

$$\lambda = -4abc/\sqrt{3}$$

Again,

$$d^2F = 2\lambda\left(\frac{dx^2}{a^2} + \frac{dy^2}{b^2} + \frac{dz^2}{c^2}\right) + 16z\,dx\,dy + 16x\,dy\,dz + 16y\,dz\,dx$$

$$= -\frac{8abc}{\sqrt{3}}\Sigma\frac{1}{a^2}dx^2 + \frac{16}{\sqrt{3}}\Sigma c\,dx\,dy \qquad \ldots(3)$$

Now from (1), we have

$$x\frac{dx}{a^2} + y\frac{dy}{b^2} + z\frac{dz}{c^2} = 0 \text{ or } \frac{dx}{a} + \frac{dy}{b} + \frac{dz}{c} = 0. \qquad \ldots(4)$$

Hence squaring,

$$\Sigma\frac{dx^2}{a^2} + 2\Sigma\frac{dx\,dy}{ab} = 0$$

or

$$abc\,\Sigma\frac{dx^2}{a^2} = -2\Sigma c\,dx\,dy$$

$$\therefore \qquad d^2F = -\frac{16}{\sqrt{3}}\,abc\,\Sigma\frac{dx^2}{a^2}$$

which is always negative.

Hence, $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$ is a point of maxima and the maximum value of $8xyz$ is $\frac{8abc}{3\sqrt{3}}$.

Note. The sign of d^2F can also be decided by expressing it in terms of dx and dy alone, be putting into (3) the value of dz from (4).

Example 12.10. Show that the length of the axes of the section of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ by the plane $lx + my + nz = 0$ are the roots of the quadratic in r^2,

$$\frac{l^2a^2}{r^2 - a^2} + \frac{m^2b^2}{r^2 - b^2} + \frac{n^2c^2}{r^2 - c^2} = 0.$$

Solution. We have to find the stationary values of the function r^2 where $r^2 = x^2 + y^2 + z^2$, subject to the two equations of condition

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad ...(1)$$

$$lx + my + nz = 0 \qquad ...(2)$$

Let us consider a function F of independent variables x, y, z,

$$F = x^2 + y^2 + z^2 + \lambda_1\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) + 2\lambda_2(lx + my + nz)$$

$$\therefore \qquad dF = 2\left(x + \frac{x\lambda_1}{a^2} + \lambda_2 l\right)dx + 2\left(y + \frac{y\lambda_1}{b^2} + \lambda_2 m\right)dy + 2\left(z + \frac{z\lambda_1}{c^2} + \lambda_2 n\right)dz$$

At stationary points,

$$x + \frac{x}{a^2}\lambda_1 + l\lambda_2 = 0,\ y + \frac{y}{b^2}\lambda_1 + m\lambda_2 = 0,\ z + \frac{z}{c^2}\lambda_1 + n\lambda_2 = 0 \qquad ...(3)$$

Multiplying by x, y, z, respectively and adding, we get

$$\lambda_1 = -(x^2 + y^2 + z^2) = -r^2$$

$$\therefore \qquad x = \frac{a^2 l\lambda_2}{r^2 - a^2},\ y = \frac{b^2 m\lambda_2}{r^2 - b^2},\ z = \frac{c^2 n\lambda_2}{r^2 - c^2}$$

But

$$0 = lx + my + nz = \lambda_2\left\{\frac{a^2 l^2}{r^2 - a^2} + \frac{b^2 m^2}{r^2 - b^2} + \frac{c^2 n^2}{r^2 - c^2}\right\}$$

and since $\lambda_2 \neq 0$, we get the quardratic in r^2 giving the stationary values:

$$\frac{a^2 l^2}{r^2 - a^2} + \frac{b^2 m^2}{r^2 - b^2} + \frac{c^2 n^2}{r^2 - c^2} = 0\,.$$

Example 12.11. If the variables x, y, z satisfy the equation

$$\phi(x)\phi(y)\phi(z) = k^3 \qquad ...(1)$$

and $\phi(a) = k \neq 0, \phi'(a) \neq 0$, show that the function

$$f(x) + f(y) + f(z) \qquad ...(2)$$

has a maximum, when $x = y = z = a$, provided that

$$f'(a)\left\{\frac{\phi''(a)}{\phi'(a)} - \frac{\phi'(a)}{\phi(a)}\right\} > f''(a).$$

Solution. Let us consider a function

$$F = f(x) + f(y) + f(z) + \lambda\{\phi(x)\phi(y)\phi(z) - k^3\}$$

$$\therefore \qquad dF = \Sigma\{f'(x) + \lambda\phi'(x)\phi(y)\phi(z)\}dx\,.$$

At stationary points,

$$\left.\begin{aligned} f'(x) + \lambda\phi'(x)\phi(y)\phi(z) &= 0 \\ f'(y) + \lambda\phi'(y)\phi(z)\phi(x) &= 0 \\ f'(z) + \lambda\phi'(z)\phi(x)\phi(y) &= 0 \end{aligned}\right\} \quad \ldots(3)$$

If the function has a maximum at (a, a, a), we have

$$f'(a) + \lambda\phi'(a)\phi(a)\phi(a) = 0$$

or

$$\lambda = -\frac{f'(a)}{\phi'(a)\phi^2(a)} = -\frac{f'(a)}{k^2\phi'(a)},\ \phi'(a) \neq 0,\ \phi(a) \neq 0$$

Again,

$$d^2F = \Sigma\left\{f''(x) + \lambda\phi''(x)\phi(y)\phi(z)\right\}dx^2 + 2\lambda\Sigma\phi'(x)\phi'(y)\phi(z)dx\,dy$$

At the stationary point (a, a, a),

$$d^2F = \{f''(a) + \lambda k^2\phi''(a)\}\ \Sigma\,dx^2 + 2\lambda k[\phi'(a)]^2\Sigma\ dx\,dy$$

From the given condition (1), we have

$$\Sigma\phi'(x)\phi(y)\phi(z)\ dx = 0$$

$\therefore$ $$k^2\phi'(a)(dx + dy + dz) = 0, \text{ at } (a, a, a)$$

or

$$dx + dy + dz = 0$$

$\therefore$ $$2\Sigma\ dx\,dy = -\Sigma\ dx^2$$

$\therefore$ $$d^2F = \{f''(a) + \lambda k^2\phi''(a)\}\ \Sigma\ dx^2 - \lambda k[\phi'(a)]^2\,\Sigma dx^2$$

$$= \left\{f''(a) - f'(a)\frac{\phi''(a)}{\phi'(a)} + f'(a)\frac{\phi'(a)}{\phi(a)}\right\}\Sigma\ dx^2$$

For a maximum value at (a, a, a), d^2F is to be negative, *i.e.*,

$$f'(a)\left\{\frac{\phi''(a)}{\phi'(a)} - \frac{\phi'(a)}{\phi(a)}\right\} > f''(a)\,.$$

Example 12.12. If $f(x, y, z) = (a^2x^2 + b^2y^2 + c^2z^2)/x^2y^2z^2$, where $ax^2 + by^2 + cz^2 = 1$, and a, b, c are positive, show that the minimum value of $f(x, y, z)$ is given by

$$x^2 = \frac{u}{2a(u + a)},\ y^2 = \frac{u}{2b(u + b)},\ z^2 = \frac{u}{2c(u + c)},$$

where u is the positive root of the equation

$$u^3 - (bc + ca + ab)\ u - 2abc = 0 \text{ (*Schlömilch*).}$$

Solution. Consider a function F of independent variables x, y, z, where

$$F = (a^2x^2 + b^2y^2 + c^2z^2)/x^2y^2z^2 + \lambda\,(ax^2 + by^2 + cz^2 - 1)$$

$$\therefore \qquad dF = \Sigma\left(2ax\lambda - \frac{2(b^2y^2 + c^2z^2)}{x^3y^2z^2}\right)dx$$

At stationary points

$$\left.\begin{aligned} 2ax\lambda - \frac{2(b^2y^2 + c^2z^2)}{x^3y^2z^2} &= 0 \\ 2by\lambda - \frac{2(c^2z^2 + a^2x^2)}{x^2y^3z^2} &= 0 \\ 2cz\lambda - \frac{2(a^2x^2 + b^2y^2)}{x^2y^2z^3} &= 0 \end{aligned}\right\}$$

or

$$\left.\begin{aligned} ax^2\lambda &= \frac{b^2y^2 + c^2z^2}{x^2y^2z^2} \\ by^2\lambda &= \frac{c^2z^2 + a^2x^2}{x^2y^2z^2} \\ cz^2\lambda &= \frac{a^2x^2 + b^2y^2}{x^2y^2z^2} \end{aligned}\right\} \qquad \ldots(1)$$

Adding,

$$\lambda = \frac{2\Sigma a^2x^2}{x^2y^2z^2} \quad (\because\ \Sigma ax^2 = 1)$$

Clearly λ is positive.

Again from (1),

$$\frac{b^2y^2 + c^2z^2}{2ax^2} = \frac{c^2z^2 + a^2x^2}{2by^2} = \frac{a^2x^2 + b^2y^2}{2cz^2} = \frac{\Sigma a^2x^2}{1}$$

or

$$\frac{a^2x^2}{1 - 2ax^2} = \frac{b^2y^2}{1 - 2by^2} = \frac{c^2z^2}{1 - 2cz^2} = \frac{\Sigma a^2x^2}{1} = \frac{u}{2}, \text{ say}$$

Clearly $u > 0$; and the coordinates of the stationary point are given by

$$\left.\begin{aligned} 2a^2x^2 = u(1 - 2ax^2) \Rightarrow x^2 &= \frac{u}{2a(u + a)} \\ \text{Similarly} \qquad y^2 = \frac{u}{2b(u + b)},\ z^2 &= \frac{u}{2c(u + c)} \end{aligned}\right\} \qquad \ldots(2)$$

Again, substituting these values in the constraint condition, $\Sigma ax^2 = 1$, we get

$$1 = \frac{u}{2(u+a)} + \frac{u}{2(u+b)} + \frac{u}{2(u+c)}$$

which on simplification shows that u is a positive root of the equation,

$$u^3 - (bc + ca + ab)\, u - 2abc = 0 \qquad ...(3)$$

Again,

$$d^2F = \Sigma 2\left(a\lambda + \frac{3(b^2y^2 + c^2z^2)}{x^4y^2z^2} \right) dx^2 + 2\Sigma \frac{4c^2}{x^3y^3} dx\, dy$$

Clearly

$$F_{xx} = 2\left(a\lambda + \frac{3(b^2y^2 + c^2z^2)}{x^4y^2z^2} \right) > 0$$

Also

$$\begin{vmatrix} F_{xx} & F_{xy} \\ F_{xy} & F_{yy} \end{vmatrix} = F_{xx}F_{yy} - (F_{xy})^2 = 4\left(a\lambda + \frac{3(b^2y^2 + c^2z^2)}{x^4y^2z^2} \right)\left(b\lambda + \frac{3(c^2z^2 + z^2x^2)}{x^2y^4z^2} \right) - \left(\frac{4c^2}{x^3y^3} \right)^2$$

Comparison of the term containing $\frac{1}{x^3y^3}$ shows that this expression is positive.

It may be similarly shown that

$$\begin{vmatrix} F_{xx} & F_{xy} & F_{zx} \\ F_{xy} & F_{yy} & F_{yz} \\ F_{zx} & F_{yz} & F_{zz} \end{vmatrix} \text{ is also positive}$$

So all the three principal minors are positive.

Thus, d^2F is positive and so the function has a minimum value at the stationary point given by (2).

EXERCISE

1. Show that

   (*i*) if $2x + 3y + 4z = a$, the maximum value of $x^2y^3z^4$ is $\left(\frac{a}{9} \right)^9$.

   (*ii*) if $a^2x^2 + 2by^3 + z^4 = c^4$, the maximum value of x^4yz^2 is given by
   $$17a^2x^2 = 12c^4,\ 17by^3 = c^4,\ 17z^4 = 3c^4.$$

2. If $xyz = abc$, the minimum value of $bcx + cay + abz$, is $3abc$.
3. If $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, the maximum value of xyz is $abc/3\sqrt{3}$.
4. If $xyz = a^2(x + y + z)$, the minimum value of $yz + zx + xy$ is $9a^2$.
5. If $x^2 + y^2 = 1$, the minimum value of $(ax^2 + by^2)/(a^2x^2 + b^2y^2)^{1/2}$ is $2(ab)^{1/2}/(a + b)$.

6. If $xyz = k^3$, the product $(x + a)\,(y + b)\,(z + c)$ is a minimum, when $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c} = \dfrac{k}{(abc)^{1/3}}$; a, b, c are positive.
7. Show that the points on the ellipse $5x^2 - 6xy + 5y^2 = 4$ for which the tangent is at the greatest distance from the origin are (1, 1) and (–1, –1).
8. Show that the point on the sphere $x^2 + y^2 + z^2 = 1$, which is farther from (2, 1, 3) is $(-2/\sqrt{14}, -1/\sqrt{14}, -3/\sqrt{14})$.
9. Show that the shortest distance from the origin to the curve of intersection of the surfaces $xyz = a$ and $y = bx$, where $a > 0, b > 0$, is $\sqrt[3]{a(b^2 + 1)/2b}$.
10. Prove that the maximum and minimum distances from the origin to the curve of intersection defined by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ and $lx + my + nz = 0$ can be obtained by solving for d the equation

$$\frac{a^2l^2}{a^2 - d^2} + \frac{b^2m^2}{b^2 - d^2} + \frac{c^2n^2}{c^2 - d^2} = 0.$$

11. If $1/x + 1/y + 1/z = 1$, show that a stationary value of $a^3x^2 + b^3y^2 + c^3z^2$ is given by $ax = by = cz$, and this gives an extreme value if $abc(a + b + c)$ is positive. (To see the sign of d^2F, change it to two variables only with the help of the condition $\Sigma l/x = 1$).
12. Find the shortest distance between the points P and Q, when P moves on the plane $x + y + z = 2a$, and Q on the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.
13. Find the maximum and the minimum distance from the origin to the curve of intersection of surface
$(x^2 + y^2 + z^2)^2 = a^2x^2 + b^2y^2 + c^2z^2$ and the plane $lx + my + nz = 0$.
14. If $lx + my + nz = 1$, l, m, n are positive constants, show that the stationary value of $xy + yz + zx$ is
$(2lm + 2mn + 2nl - l^2 - m^2 - n^2)^{-1}$
and that this value is a maximum when it is positive.
15. Show that the function $xy + yz + zx$ has no extreme value, but has a maximum value when x, y, z are constrained by the condition $ax + by + cz = 1$, where a, b, c are positive constants satisfying the condition
$2(ab + bc + ca) > (a^2 + b^2 + c^2)$.
16. If $\phi(x, y) = 0$, show that the determinant

$$\begin{vmatrix} f_{xx} + \lambda\phi_{xx} & f_{xy} + \lambda\phi_{xy} & \phi_x \\ f_{xy} + \lambda\phi_{xy} & f_{yy} + \lambda\phi_{yy} & \phi_y \\ \phi_x & \phi_y & 0 \end{vmatrix}$$

where λ is Lagrange's multiplier, is positive, in case the function $f(x, y)$ attains a maximum.
17. If $ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy$ is equal to a constant k, and $lx + my + nz = 0$, find the maximum and minimum values of $x^2 + y^2 + z^2$.
18. If $ax + by + cz = 1$, show that in general $x^3 + y^3 + z^3 - 3xyz$ has the two stationary values 0 and $(a^3 + b^3 + c^3 - 3abc)^{-1}$, of which the first is a maximum or a minimum according as $a + b + c \gtrless 0$, but the second is not an extreme value. Discuss in particular, the cases when (*i*) $a + b + c = 0$, (*ii*) $a = b = c$.
19. If $x + y + z = 1$, find the stationary values of
$x^3 + y^3 + z^3 + 3mxyz, \ (m \neq 2)$
and show that the symmetrical stationary value is a maximum or minimum according as $m \lessgtr 2$, but the other stationary values are not extreme values.

Also show that $x^3 + y^3 + z^3 + 6xyz$ has only one stationary value and no extreme value.

APPENDIX I

THEOREMS ON REARRANGEMENT OF TERMS AND TESTS FOR ARBITRARY SERIES

We propose to discuss first some tests for arbitrary term series and then some theorems on rearrangement of terms of such series.

1. TESTS FOR ARBITRARY TERM SERIES

Let us first prove a lemma.

Lemma

If b_n is a positive, monotonic decreasing function and if A_n is bounded, then the series $\sum A_n(b_n - b_{n+1})$, is absolutely convergent.

Since A_n is bounded, therefore $\exists$ is a positive number k such that

$$|A_n| < k, \text{ for all } n.$$

Thus
$$\sum_{n=1}^{m} |A_n(b_n - b_{n+1})| = \sum_{n=1}^{m} A_n(b_n - b_{n+1}), \qquad [\because b_n - b_{n+1} \geq 0]$$

$$< k \sum_{n=1}^{m} (b_n - b_{n+1})$$

$$= k(b_1 - b_{m+1})$$

$$< k\, b_1.$$

Thus the sequence of partial sums of the positive term series $\sum |A_n(b_n - b_{n+1})|$ is bounded above by kb_1, so that the series $\sum |A_n(b_n - b_{n+1})|$ converges, *i.e.* the series $\sum A_n(b_n - b_{n+1})$ converges absolutely.

Note. The LEMMA may be restated as follows:
If $\{b_n\}$ is a positive, monotonic decreasing sequence and if $\{A_n\}$ is a bounded sequence, then the series $\sum A_n(b_n - b_{n+1})$ is absolutely convergent.

1.1 Abel's Test

If b_n is a positive monotonic decreasing function and if $\sum u_n$ is a convergent series, then the series $\sum u_n b_n$ is also convergent.

Let $v_n = u_n b_n,$

and $S_n = \sum_{r=1}^{n} u_r, V_n = \sum_{r=1}^{n} v_r,$ be n^{th} partial sums.

Then
$$\begin{aligned} V_n &= u_1b_1 + u_2b_2 + \cdots + u_nb_n \\ &= S_1b_1 + (S_2 - S_1)b_2 + \cdots + (S_n - S_{n-1})b_n \\ &= S_1(b_1 - b_2) + S_2(b_2 - b_3) + \cdots + S_{n-1}(b_{n-1} - b_n) + S_nb_n \\ &= \sum_{r=1}^{n-1} S_r(b_r - b_{r+1}) + S_nb_n. \end{aligned} \quad ...(1)$$

Since the series $\sum u_n$ is convergent, therefore the sequence $\{S_n\}$ is also convergent and hence bounded. Also b_n is a positive and monotonic decreasing function. Therefore by the above Lemma, the series $\sum S_n(b_n - b_{n+1})$ and, hence the partial sum $\sum_{r=1}^{n-1} S_r(b_r - b_{r+1})$ tends to a finite limit as $n \to \infty$.

Also since $\{b_n\}$ is monotonic decreasing and bounded below by zero, therefore $\{b_n\}$ is convergent and hence b_n tends to a finite limit as $n \to \infty$. Hence S_nb_n tends to a finite limit as $n \to \infty$.

Using the above results we find from (1) that V_n tends to a finite limit as $n \to \infty$ *i.e.*, the sequence $\{V_n\}$ of partial sums of $\sum v_n,$ converges. Consequently the series $\sum v_n$ or $\sum u_n b_n$ converges.

Corollary. A convergent series $\sum u_n$ (which need not convergent absolutely) remains convergent if its terms are each multiplied by a factor a_n, provided that the sequence $\{a_n\}$ is bounded and monotonic.

Under the given conditions, $\{a_n\}$ converges to a limit, a. Let us write $b_n = a - a_n$ when $\{a_n\}$ is an increasing sequence, and $b_n = a_n - a$ when $\{a_n\}$ is decreasing. Then it is clear that the sequence $\{b_n\}$ monotonically decreases to the limit zero. With this function b_n, we deduce, as above that the series $\sum u_nb_n$ converges.

Also since $\sum u_n$ and hence $\sum au_n$ converges, the convergence of $\sum u_na_n$ follows.

1.2 Dirichlet's Test

If b_n is a positive, monotonic decreasing function with limit zero, and if, for the series $\sum u_n$, the sequence $\{S_n\}$ of partial sums is bounded, then the series $\sum u_nb_n$ is convergent.

Using the notation of §1.1, we get as before.

$$V_n = \sum_{r=1}^{n-1} S_r(b_r - b_{r+1}) + S_nb_n \quad ...(1)$$

Since S_n is bounded and since b_n is positive and monotonic decreasing, therefore by the above Lemma, $\sum_{r=1}^{n-1} S_r(b_r - b_{r+1})$ tends to a finite limit as $n \to \infty$.

Also since $b_n \to 0$ as $n \to \infty$ and since S_n is bounded, therefore $S_n b_n \to 0$ as $n \to \infty$.

Using the above results we find from (1) that V_n tends to a finite limit as $n \to \infty$ and hence the series $\sum v_n$ $(= \sum u_n b_n)$ converges.

The case $u_n = (-1)^{n-1}$ of the above theorem is of considerable importance. In fact we deduce.

Corollary. Leibnitz test as a particular case of Dirichlet's text.

Since the sequence of partial sums of the series $\sum(-1)^{n-1}$ is bounded (for $S_n = 0$ if n is even, and $S_n = 1$ if n is odd) therefore by taking $u_n = (-1)^{n-1}$, $\sum u_n b_n$ reduces to $b_1 - b_2 + b_3 - b_4 + ...$ Thus we obtain, 'if b_n is positive and monotonic decreasing to the limit zero, then the alternating series $b_1 - b_2 + b_3 - b_4 + ...$ is convergent.

Example 1. Show that the series

$$0 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{3} + \frac{2}{3^2} - \frac{1}{4} + \frac{3}{4^2} - ... \text{ converges.}$$

Solution. The given series can be considered to have arisen as a result of multiplication of the terms of the series.

$$1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + ... \qquad ...(A)$$

by the terms of the sequence

$$0, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}, \frac{2}{3}, \frac{3}{4}, \frac{3}{4}, ..., \qquad ...(B)$$

Since the series (A) is convergent and the sequence (B) is monotonic and bounded, therefore by Abel's tests the given series converges.

Example 2. Test the convergence of the series

$$\sum \frac{(n^3 + 1)^{1/3} - n}{\log n}.$$

Solution. Let $u_n = \{(n^3 + 1)^{1/3} - n\}$,

and $b_n = \dfrac{1}{\log n}$.

Then the given series can be written as $\sum b_n u_n$.

Since $\sum u_n$ converges and $\{b_n\}$ is a positive monotonic decreasing sequence, therefore by Abel's test, the given series converges.

Example 3. Show that the series

$$1-\frac{1}{3.2^2}+\frac{1}{5.3^2}-\frac{1}{7.4^2}+\ldots, \text{ is convergent.}$$

Solution. Let $\sum u_n=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots,$

and $$b_n=\frac{1}{2n-1}.$$

Then the given series can be written as $\sum b_n u_n$.

Since $\sum u_n$ converges and b_n is positive and monotonic decreasing, therefore by Abel's test, the given series converges.

2. REARRANGEMENT OF TERMS

It is a well known fact, that a finite sum keeps the same value, no matter how the terms of the sum are arranged. This property, however, is by no means universally true for infinite series. For example the series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots,$$

converges to a sum, say S. On rearranging the terms so that each positive term is followed by two negative terms, the series

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\ldots,$$

converges to the sum $\frac{1}{2}$ S.

Another arrangement, gives the series (with sign changed)

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2^8}-1+\frac{1}{2^8}+\ldots+\frac{1}{2^{16}}-\frac{1}{3}+\ldots,$$

which is divergent.

Consequently, the *rearrangement* (or equally well *derangement*) or changing the order of the terms in a series may not only alter the sum of the series but may change its nature all together. So we naturally ask *under what conditions may we rearrange the terms of the series without altering its value*?

A series $\sum v_n$ is said to arise from a series $\sum u_n$ by a rearrangement of terms if there exists a one-to-one correspondence between the terms of the two series, so that every term in the first series occupies a perfectly definite place in the second series, and conversely. Thus, corresponding to any number of terms (say n) in the first series, we can find a number p such that p terms of the second series contain all the n terms (and some others) of the first; and conversely.

2.1 Theorem 1. (*Dirichlet's theorem*). *A series obtained from an absolutely convergent series by a rearrangement of terms converges absolutely and has the same sum as the original series.*

We shall prove the theorem in two parts, first for the positive terms series and then for series of arbitrary terms.

Let $\sum u_n$ be a given series of positive terms which converges to a sum, say S. $\sum v_n$ be the rearranged series. Let S_m and σ_m respectively denoted the partial sums of the series $\sum u_n$ and $\sum v_n$.

As σ_m consists of m terms of the series $\sum v_n$, we can find a number p such that p terms of the series $\sum u_n$ contain all the m terms (and some more) of the former and since we are dealing with positive terms series, therefore $\sigma_m \leq S_p \leq S$.

Thus, the sequence $\{\sigma_m\}$ of partial sums of the series $\sum v_n$ of positive terms is bounded above by S. Therefore, the sequence $\{\sigma_m\}$ and consequently the series $\sum v_n$ converges to a limit, say σ, where $\sigma \leq S$.

Considering $\sum u_n$ as rearrangement of $\sum v_n$, we can similarly prove that $S \leq \sigma$.

Hence $\sigma = S$, *i.e.* the two series converge to the same limit.

Let now $\sum u_n$ be an absolutely convergent series of arbitrary terms and $\sum v_n$ a rearrangement of $\sum u_n$.

Let

$$a_n = \begin{cases} u_n, & \text{if } u_n \geq 0, \\ 0, & \text{if } u_n \leq 0; \end{cases}$$

$$b_n = \begin{cases} -u_n, & \text{if } u_n \leq 0, \\ 0, & \text{if } u_n \geq 0; \end{cases}$$

$$a_n' = \begin{cases} v_n, & \text{if } v_n \geq 0, \\ 0, & \text{if } v_n \leq 0; \end{cases}$$

$$b_n' = \begin{cases} -v_n, & \text{if } v_n \leq 0, \\ 0, & \text{if } v_n \geq 0. \end{cases}$$

Then clearly a_n, b_n, a_n', b_n' are non-negative, and

$$u_n = a_n - b_n, |u_n| = a_n + b_n$$

$$v_n = a_n' - b_n', |v_n| = a_n' + b_n' \qquad \ldots(i)$$

$$\Rightarrow \quad \left.\begin{aligned} a_n &= \frac{1}{2}(|u_n| + u_n) \\ b_n &= \frac{1}{2}(|u_n| - u_n) \end{aligned}\right\} \qquad \ldots(ii)$$

and

$$\left.\begin{aligned} a_n' &= \frac{1}{2}(|v_n| + v_n) \\ b_n' &= \frac{1}{2}(|v_n| - v_n) \end{aligned}\right\} \qquad \ldots(iii)$$

Since Σu_n is absolutely convergent, therefore from (*ii*) Σa_n and Σb_n are also convergent (*Ref.* Theorem 5). Again since Σa_n and Σb_n are convergent series of non-negative terms and $\Sigma a_n'$ and $\Sigma b_n'$ are respectively their rearrangements, therefore by what has been proved above, $\Sigma a_n'$ and $\Sigma b_n'$ are also convergent. Also if a, b, a', b' denoted the sum of the series $\Sigma a_n, \Sigma b_n, \Sigma a_n', \Sigma b_n'$ respectively, then $a = a'$ and $b = b'$.

From (*i*) it follows at once that Σv_n and $\Sigma |v_n|$ are convergent and

$$\sum_{n=1}^{\infty} v_n = a' - b' = a - b = \sum_{n=1}^{\infty} u_n$$

$$\sum_{n=1}^{\infty} |v_n| = a' + b' = a + b = \sum_{n=1}^{\infty} |u_n|$$

Hence, the rearranged series Σv_n converges absolutely to the same sum as Σu_n.

Remarks **1.** For positive term series the theorem may be stated as follows:
A series of positive terms, if convergent, has a sum independent of the order of its terms, but if divergent, it remains divergent however its terms are rearranged.
For the divergent case, one may argue as follows:
If Σu_n is divergent, Σv_n cannot converge; for the foregoing argument shows that if Σv_n converges, Σu_n (regarded as a rearrangement of Σv_n) must also converge. Consequently Σv_n is divergent.

2. The above theorem (along with theorem 5) shows that the brackets may be inserted or removed, or terms be picked and placed at random without changing the behaviour of a positive term series or an absolutely convergent series. In fact they behave exactly like finite sums.

3. An absolutely convergent series because it remains convergent, with unaltered sum under every rearrangement of terms is also called *unconditionally convergent*.

2.2 We shall now prove a theorem which though of no practical importance, is of considerable theoretical interest.

Theorem 2. (*Riemann's Theorem*).

By an appropriate rearrangement of the terms, a conditionally convergent series Σu_n *can be made*

(*i*) *to converge to any number* σ,
(*ii*) *to diverge to* $+\infty$, *or*
(*iii*) *to diverge to* $-\infty$, *or*
(*iv*) *to oscillate finitely, or*
(*v*) *to oscillate infinitely.*

Let $a_n = \begin{cases} u_n, & \text{if } u_n \geq 0 \\ 0, & \text{if } u_n < 0 \end{cases}$;

and $b_n = \begin{cases} -u_n, & \text{if } u_n \leq 0, \\ 0, & \text{if } u_n > 0. \end{cases}$

Then clearly a_n, b_n are non-negative, and

$$u_n = a_n - b_n, \ |u_n| = a_n + b_n. \qquad ...(1)$$

Since $\sum u_n$ is conditionally convergent, therefore $\sum |u_n|$ diverges and hence from (1) at least one of the series $\sum a_n, \sum b_n$ diverges.

Again from (1) since $\sum u_n$ converges therefore the two series $\sum a_n, \sum b_n$ either both converge or both diverge (being non-negative term series, they cannot oscillate). Thus we infer that $\sum a_n, \sum b_n$ both diverge.

Also $$a_n \to 0, b_n \to 0 \text{ as } n \to \infty. \qquad (\because u_n \to 0)$$

(*i*) We shall first show that a rearrangement $\sum v_n$ of $\sum u_n$ can be found which converges to any number, σ.

Let n_1 be the least number of terms of $\sum a_n$ whose sum

$$a_1 + a_2 + a_3 + ... + a_{n_1} > \sigma.$$

Let m_1 be the least number of terms of $\sum b_n$ such that

$$a_1 + a_2 + a_3 + ... + a_{n_1} - b_1 - b_2 - ... - b_{m_1} < \sigma.$$

Again let n_2 be least number of the next terms (terms following a_{n_1}) of $\sum a_n$ such that

$$a_1 + a_2 + ... + a_{n_1} - b_1 - b_2 - ... - b_{m_1} + a_{n_1+1} + a_{n_1+2} + ... + a_{n_1+n_2} > \sigma.$$

Let m_2 be the least number of the next terms of $\sum b_n$ such that when $(-b_{m_1+1} - b_{m_1+2} - ... - b_{m_1+2})$ added to the above sum, makes it less than σ. The process may be continued indefinitely. The process indicated above is always possible because of the divergence of the two series $\sum a_n, \sum b_n$.

Let $\sum v_n$ be the rearranged series and $\{\sigma_n\}$ its sequence of partial sums.

Clearly $$\sigma_{n_1} > \sigma, \sigma_{n_1+m_1} < \sigma, \sigma_{n_1+m_1+n_2} > \sigma, ...$$

Therefore it can be easily shown that the sequence $\{\sigma_n\}$ converges to σ.

$\Rightarrow$ the rearrangement $\sum v_n$ converges to σ.

(*ii*) We shall now show that a suitable rearrangement of $\sum u_n$ can be found which diverges to $+\infty$.

Let us consider the rearrangement

$$a_1 + a_2 + ... + a_{m_1} - b_1 + a_{m_1+1} + a_{m_1+2} + ... + a_{m_2} - b_2 + a_{m_2+1} + ...,$$

in which a group of positive terms is followed by a single negative terms. This is certainly a rearrangement of $\sum u_n$ and let us denote it by $\sum v_n$ and its partial sums by S_n.

Now since the series $\sum a_n$ is divergent, its partial sums are therefore unbounded.

Let us first choose m_1 so large that

$$a_1 + a_2 + a_3 + ... + a_{m_1} > 1 + b_1,$$

then $m_2 > m_1$ so large that

$$a_1 + a_2 + ... + a_{m_1} + a_{m_1+1} + ... + a_{m_2} > 2 + b_1 + b_2$$

and generally, $m_n > m_{n-1}$ so large that

$$a_1 + a_2 + ... + a_{m_n} > n + b_1 + b_2 + ... + b_n,$$

where $n = 1, 2, 3, ...$

Now since each of the partial sum $S_{m_1+1}, S_{m_2+1}, \cdots$ of $\sum v_n$ whose last terms is a negative terms $-b_n$, is greater than n ($n = 1, 2, 3, ...$), therefore these partial sums are unbounded above and consequently the series $\sum v_n$ diverges to $+\infty$.

(*iii*) By considering the rearrangement

$$-b_1 - b_2 - ... - b_{m_1} + a_1 - b_{m_1+1} - b_{m_1+2} - ... - b_{m_2} + a_2 - b_{m_1+1} - ...,$$

it can be shown, as before, that the rearrangement diverges to $-\infty$.

Other cases may similarly be proved by considering suitable rearrangements of the given series.

Remark. As proved earlier the absolutely convergent series remain convergent, with unaltered sum, without any condition on rearrangement of terms but it is not so in the case of non-absolutely convergent series (*i.e.* series which converge but not absolutely). Such series change their behavior by change in the order of the terms. This is precisely the reason why such series are called *conditionally convergent*.

Example 4. Criticise the following paradox:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + ...$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + ... - 2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + ...\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + ... - \left(1 + \frac{1}{2} + \frac{1}{3} + ...\right)$$

$$= 0.$$

Hence, the series converges to zero.

The given series is conditionally convergent and hence can be made to converge to any limit by a rearrangement of the terms.

EXERCISE

1. Assuming the convergence of $\sum u_n$, show that the following series are convergent:

(*i*) $\sum_n^{u_n}$ (*ii*) $\sum_{\log n}^{u_n}$ (*iii*) $\sum \frac{u_n}{\log_a n}$

(*iv*) $\sum\left(\frac{n+1}{n}\right)u_n$ (*v*) $n^{1-n}\, u_n$ (*vi*) $\sum\left(1 + \frac{1}{n}\right)^n u_n$.

2. Examine the convergence of the series:

(*i*) $\sum_{n^\alpha}^{\cos n\theta}$, (*ii*) $\sum \frac{\sin n^\alpha}{n^\alpha}$.

[*Hint:*

$$S_n = \sum_{r=1}^{n} \cos r^{\theta} = \frac{\cos \frac{n-1}{2}\theta \sin \frac{n^{\theta}}{2}}{\sin \frac{\theta}{2}}, \theta \neq 0, 2k\pi$$

$$S_n = \sum_{r=1}^{n} \sin r^{\theta} = \frac{\sin \frac{n-1}{2}\theta \sin \frac{n^{\theta}}{2}}{\sin \frac{\theta}{2}}, \theta \neq 0, 2k\pi.$$

Thus S_n is a bounded function *i.e.* the series $\Sigma \cos n^{\theta}$ or $\Sigma \sin n^{\theta}$ is such that its partial sums are bounded, when θ is neither zero nor a multiple of 2π. By Dirichlet's test, the series converges for $\alpha > 0$. For $\alpha \leq 0$ since the nth term does not tend to zero, both the series diverge.

For $\theta = 0$ or $2k\pi$, the series $\Sigma \frac{\cos n^{\theta}}{n^{\alpha}}$ reduces to $\Sigma \frac{1}{n^{\alpha}}$ which is convergent for $\alpha > 1$; the series $\Sigma \frac{\sin n^{\theta}}{n^{\alpha}}$ reduces to series of zeros which is evidently convergent for all value of α.]

APPENDIX II

CANTOR'S THEORY OF REAL NUMBERS

1. INTRODUCTION

In an earlier Chapter 1, we followed Dedekind's theory to extend the rational number system to the real number system. We now introduce another method, which is less algebraic but more sophisticated, due to G. Cantor. The construction of real numbers by Cantor's theory depends on the existence of an equivalence relation in the set of all Cauchy sequences of rational numbers. Defining each equivalence class as a real number and by suitably defining addition and multiplication, and order, this set **R** of Cantor's real numbers will be made into an ordered field which will be an extension of the ordered field **Q** of rational numbers. It will then be shown that this ordered field **R** is also *complete* (in the sense defined earlier) i.e., the field **R** is *order complete*.

Since the concept of *Cauchy sequences of rational numbers* and their limits are basic to Cantor's Theory, we start the discussion by a brief introduction to these sequences.

2. SEQUENCES OF RATIONAL NUMBERS

Definitions

1. A function S on the set **N** of natural numbers into the set **Q** of rational numbers is called a *rational sequence* or a *sequence of rational numbers* and is symbolically denoted as $\mathbf{S} : \mathbf{N} \to \mathbf{Q}$.
2. A rational sequence $\{S_n\}$ is said to be *bounded* if there exists a rational numbers $K > 0$, such that
$$| S_n | \leq K, \quad \forall\ n$$
3. A sequence $\{S_n\}$ of rational numbers is called a *Cauchy sequence* or a *fundamental sequence*, if for each rational number $\varepsilon > 0$, there exists a positive integer m_0, such that
$$| S_n - S_m | < \varepsilon, \quad \forall\ m, n \geq m_0$$
or
$$| S_{n+p} - S_n | < \varepsilon, \quad \forall\ n \geq m_0, \text{ integer } p \geq 1$$
4. A rational sequence $\{S_n\}$ is said to *converge* to a rational number l (or we have the number l as its *limit*) if for each rational number $\varepsilon > 0$, there exists a positive integer m_0, (depending on ε), such that $| S_n - l | < \varepsilon$, for all $n \geq m_0$, and we then write
$$\lim_{n \to \infty} S_n = l \text{ or } \quad S_n \to l$$

It may be easily shown that

(*i*) every Caucy sequence is bounded,

(*ii*) every convergent sequence is bounded, and

(*iii*) every convergent sequence is a Cauchy sequence, i.e., a necessary condition for convergence of a rational sequence is that it is a Cauchy sequence.

Thus, a *necessary condition for convergence* of a rational sequence is that for any rational number $\varepsilon > 0$, there exists a positive integer m_0, such that

$$| S_{n+p} - S_n | < \varepsilon, \quad \forall\ n \geq m_0,\ p \geq 1 \qquad ...(1)$$

Example. Show that the sequence $\{S_n\}$, where

$$S_{n+1} = \frac{2 + S_n}{1 + S_n}, \quad n \geq 1$$

$$S_1 = 1$$

is not convergent in the field of rational numbers.

Solution. If the sequence converges to a rational number, say q, then

$$\lim S_{n+1} = q = \lim S_n$$

$$\therefore \qquad q = \lim \frac{2 + S_n}{1 + S_n} = \frac{2 + q}{1 + q}$$

or

$$q^2 = 2$$

But no rational number exists whose square is equal to 2. Hence, the sequence is not convergent in the field of rational numbers.

Note. $\{S_n\}$ is the rational sequence, $1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \ldots$

Thus by considering the sequence $\{S_n\}$, where $S_{n+1} = \frac{2 + S_n}{1 + S_n}$, for $n \geq 1$, $S_1 = 1$, it may be shown that (1) is not a sufficient condition for convergence of a rational sequence. Thus, every Cauchy sequence is not convergent in the field of rationals (compare § 3.6.1, Ch. 3)

EXERCISE

1. Show that the sequence $\{S_n\}$, where $S_{n+1} = \frac{l + 2 + kS_n}{k + lS_n}$, $S_1 = 1$, l, k are finite non-zero numbers, is not convergent in the field of rationals.

2. If $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences of rational numbers, then

   (*i*) the sequences $\{a_n \pm b_n\}, \{a_n b_n\}$ are also Cauchy sequences,

   (*ii*) the sequences $\left\{\frac{1}{b_n}\right\}$ and $\left\{\frac{a_n}{b_n}\right\}$ are also Cauchy sequences provided $\{b_n\}$ does not converge to zero, and $b_n \neq 0$ for any n.

3. REAL NUMBERS

We shall use F_Q to denote the set of all Cauchy sequences of rational numbers.

Definition. A sequence $\{a_n\} \in F_Q$ *is said to be equivalent to* $\{b_n\} \in F_Q$ *whenever* $\{a_n - b_n\}$ *converges to zero*. Expressed symbolically,

$$\{a_n\} \sim \{b_n\} \text{ iff } \lim (a_n - b_n) = 0$$

Theorem 1. *The relation ~ in the set of all Cauchy sequences of rational numbers, defined by*

$$\{a_n\} \sim \{b_n\} \text{ iff } \lim (a_n - b_n) = 0$$

is an equivalence relation.

(1) For any Cauchy sequence $\{a_n\} \in F_Q$

$$\lim (a_n - a_n) = 0$$

$$\therefore \quad \{a_n\} \sim \{a_n\}$$

and so the relation ~ is reflexive.

(2) Let $\{a_n\}, \{b_n\}$ be sequences in F_Q, such that $\lim (a_n - b_n) = 0$.
Then,

$$\lim (b_n - a_n) = \lim [-(a_n - b_n)] = 0$$

$$\therefore \quad \{a_n\} \sim \{b_n\} \Rightarrow \{b_n\} \sim \{a_n\}$$

and so the relation ~ is symmetric.

(3) Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences in F_Q, such that

$$\{a_n\} \sim \{b_n\} \text{ and } \{b_n\} \sim \{c_n\}$$

i.e.,

$$\lim (a_n - b_n) = 0 \text{ and } \lim (b_n - c_n) = 0$$

Then,

$$\lim (a_n - c_n) = \lim (a_n - b_n + b_n - c_n)$$
$$= \lim (a_n - b_n) + \lim (b_n - c_n) = 0$$

$$\therefore \quad \{a_n\} \sim \{b_n\} \wedge \{b_n\} \sim \{c_n\} \Rightarrow \{a_n\} \sim \{c_n\}$$

Hence, the relation ~ is transitive.

Hence, from (1), (2) and (3) it follows that the relation ~ is an equivalence relation in F_Q.

Notation. Let $[a_n]$ denote the equivalence class containing the sequence $\{a_n\}$, i.e., the set of rational Cauchy sequences equivalent to $\{a_n\}$. Thus

$$[a_n] = \{\{x_n\} \in F_Q \mid \{x_n\} \sim \{a_n\}\}$$

EXERCISE

1. If $\{a_n\} \in F_Q$, then $\lim (a_n) = a$ iff $\{a_n\} \sim \{a\}$.
2. Two equivalence classes $[a_n]$, and $[a'_n]$ are equal iff $\{a_n\} \sim \{a'_n\}$.

Definition. A Cantor *real number* is an equivalence class $[a_n]$ with respect to the equivalence relation $\sim$ in F_Q defined by the condition

$$\{a_n\} \sim \{b_n\} \text{ iff } \lim (a_n - b_n) = 0.$$

Thus, if ξ is a real number in $[a_n]$, then

$$\xi = \{\{x_n\} \in F_Q \mid \{x_n\} \sim \{a_n\}\}.$$

We shall denote by $\mathbf{R}$ the set of all real numbers and use $\xi, \eta, \ldots$ to denote the real numbers.

4. ADDITION AND MULTIPLICATION IN R

We shall define two binary operations (+ and .) in $\mathbf{R}$, to be called addition and multiplication and discuss some of their properties. But we shall need the following theorems before we can do so.

Theorem 2. *If* $\{x_n\}$, $\{y_n\}$, $\{a_n\}$, *and* $\{b_n\}$ *belong to* F_Q, *such that* $\{x_n\} \sim \{a_n\}$ *and* $\{y_n\} \sim \{b_n\}$, *then* $\{x_n + y_n\}$ *and* $\{a_n + b_n\}$ *also belong to* F_Q *and* $\{x_n + y_n\} \sim \{a_n + b_n\}$.

Since $\{x_n\}$, $\{y_n\}$, $\{a_n\}$ and $\{b_n\}$ all belong to F_Q, therefore as in Exercise 2 § 1, $\{x_n + y_n\}$ and $\{a_n + b_n\}$ also belong to F_Q. Also

$$\{x_n\} \sim \{a_n\} \Rightarrow \lim (x_n - a_n) = 0$$

and

$$\{y_n\} \sim \{b_n\} \Rightarrow \lim (y_n - b_n) = 0$$

$$\therefore \quad \lim (x_n + y_n - a_n - b_n) = \lim (x_n - a_n) + \lim (y_n - b_n) = 0$$

$$\Rightarrow \quad \{x_n + y_n\} \sim \{a_n + b_n\}.$$

Theorem 3. *If* $\{x_n\}$, $\{y_n\}$, $\{a_n\}$, *and* $\{b_n\}$ *belong to* F_Q, *such that* $\{x_n\} \sim \{a_n\}$ *and* $\{y_n\} \sim \{b_n\}$, *then* $\{x_n y_n\}$ *and* $\{a_n b_n\}$ *also belong to* F_Q *and* $\{x_n y_n\} \sim \{a_n b_n\}$.

Since $\{x_n\}$, $\{y_n\}$, $\{a_n\}$ and $\{b_n\}$ all belong to F_Q, therefore, $\{x_n y_n\}$ and $\{a_n b_n\}$ also belong to F_Q.

Again, since $\{x_n\}, \{b_n\}$ are rational Cauchy sequences, they are bounded and therefore, $\exists$ positive rational numbers k_1, k_2, such that

$$|x_n| < k_1, |b_n| < k_2, \quad \forall\ n \in \mathbf{N}$$

Also since $\{x_n\} \sim \{a_n\}$ and $\{y_n\} \sim \{b_n\}$,

$$\therefore \quad \lim (x_n - a_n) = 0 \text{ and } \lim (y_n - b_n) = 0$$

so that for rational $\varepsilon > 0, \exists$ positive integers m_1, m_2 that

$$|x_n - a_n| < \varepsilon/2k_2, \quad \text{for } n \geq m_1$$

and

$$|y_n - b_n| < \varepsilon/2k_1, \quad \text{for } n \geq m_2$$

Let $m = \max(m_1, m_2)$.

Hence, for $n \geq m$, we have

$$|x_n y_n - a_n b_n| = |x_n(y_n - b_n) + b_n(x_n - a_n)|$$

$$\leq |x_n||y_n - b_n| + |b_n||x_n - a_n|$$

$$< k_1 \frac{\varepsilon}{2k_1} + k_2 \frac{\varepsilon}{2k_2} = \varepsilon.$$

$$\Rightarrow \quad \lim (x_n y_n - a_n b_n) = 0$$

$$\therefore \quad \{x_n y_n\} \sim \{a_n b_n\}.$$

Theorem 4. (*Addition in* **R**). *There is a binary operation* $f: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, *such that for every pair of real numbers* $\xi = [a_n]$, $\eta = [b_n]$,

$$f(\xi, \eta) = \zeta$$

where

$$\zeta = [a_n + b_n]$$

Let (ξ, η) be an arbitrary element of $\mathbf{R} \times \mathbf{R}$ and let $\xi = [a_n]$, $\eta = [b_n]$, for some sequences $\{a_n\}$ and $\{b_n\}$ in F_Q.

Since $\{a_n + b_n\} \in F_Q$, therefore, $[a_n + b_n] \in \mathbf{R}$.

Let $f(\xi, \eta) = \zeta$, where $\zeta = [a_n + b_n]$.

We now show that mapping f is well defined.

Let, if possible, $f(\xi, \eta) = \zeta'$, where $\zeta' = [a_n' + b_n']$, and $\xi = [a_n']$, $\eta = [b_n']$.

Now

$$[a_n] = \xi = [a_n'] \text{ and } \eta = [b_n] = [b_n']$$

$$\therefore \quad \{a_n\} \sim \{a_n'\} \text{ and } \{b_n\} \sim \{b_n'\}$$

$$\Rightarrow \quad \{a_n + b_n\} \sim \{a_n' + b_n'\}$$

$$\Rightarrow \quad [a_n + b_n] = [a_n' + b_n']$$

$$\Rightarrow \quad \zeta = \zeta'$$

Thus, f is a binary operation on **R**.

Definition. The binary operation f on **R** is called *addition* in **R** and is denoted by +. Thus

For every pair ξ, η *of real numbers where* $\xi = [a_n]$ *and* $\eta = [b_n]$, *for some sequences* $\{a_n\}$ *and* $\{b_n\}$ *in* F_Q, *the real number* $\zeta = [a_n + b_n]$ *is called the sum of* ξ *and* η *and is denoted by* $\xi + \eta$.

EXERCISE

1. For real numbers $\xi, \eta, \zeta, \ldots$, prove that

   (*i*) $\xi + \eta = \eta + \xi$

(*ii*) $\xi + (\eta + \zeta) = (\xi + \eta) + \zeta$

(*iii*) There is a unique real number 0, such that

$$\xi + 0 = 0 + \xi = \xi$$

[*Hint*. Take $0 = [0_n]$, where $0_n = 0$ for all n.]

(*iv*) For every $\xi \in \mathbf{R}$, there exists a unique $\eta \in \mathbf{R}$, such that

$$\xi + \eta = 0 = \eta + \xi.$$

Theorem 5. (**Multiplication** *in* **R**). *There is a binary operation* $g: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, *such that for every pair of real numbers* $\xi = [a_n]$, $\eta = [b_n]$,

$$g(\xi, \eta) = \zeta,$$

where $\zeta = [a_n b_n]$.

Let (ξ, η) be an arbitrary element of $\mathbf{R} \times \mathbf{R}$ and let $\xi = [a_n]$, $\eta = [b_n]$, for some sequences $\{a_n\}$ and $\{b_n\}$ in F_Q.

Since $\{a_n b_n\} \in F_Q$, therefore, $[a_n b_n] \in \mathbf{R}$.

Let $g(\xi, \eta) = \zeta$, where $\zeta = [a_n b_n]$.

We now show that the mapping g is well defined.

Let, if possible, $g(\xi, \eta) = \zeta'$, where $\zeta' = [a_n' b_n']$, and $\xi = [a_n']$, $\eta = [b_n']$.

Now

$$[a_n] = [a_n'], \text{ and } [b_n] = [b_n']$$

$$\therefore \quad \{a_n\} \sim \{a_n'\} \text{ and } \{b_n\} \sim \{b_n'\}$$

$$\Rightarrow \quad \{a_n b_n\} \sim \{a_n' b_n'\}$$

$$\Rightarrow \quad [a_n b_n] = [a_n' b_n']$$

$$\Rightarrow \quad \zeta = \zeta'$$

Thus g is a binary operation on **R**.

Definition. The binary operation g on **R** is called *multiplication* in **R** and is denoted by '.'. Thus for every pair ξ, η or real numbers where $\xi = [a_n]$ and $\eta = [b_n]$ for some sequences $\{a_n\}$ and $\{b_n\}$ in F_Q, the real number $\zeta = [a_n b_n]$ is called the product of ξ and η and is denoted by $\xi \cdot \eta$.

EXERCISE

1. For real numbers $\xi, \eta, \zeta, \ldots$, show that

(*i*) $\xi \cdot \eta = \eta \cdot \xi$

(*ii*) $\xi \cdot (\eta \cdot \zeta) = (\xi \cdot \eta) \cdot \zeta$,

(*iii*) There exists a real number 1, called the multiplicative identity such that

$$\xi \cdot 1 = 1 \cdot \xi = \xi$$

[*Hint*. Take $1 = [I_n]$, where $I_n = 1$ for all n.]

(*iv*) The multiplicative identity is unique.

(*v*) To each real number $\xi \neq 0$, there corresponds a unique real number η, such that

$$\xi \cdot \eta = 1 = \eta \cdot \xi$$

[Let $\xi = [a_n]$, Since $\xi \neq 0$, the sequence $\{a_n\}$ can have at the most a finite number of terms equal to zero. Let $m \in \mathbf{N}$ be such that $a_n \neq 0$ for $n > m$.

Let us define a sequence $\{b_n\}$, where $b_n = \dfrac{1}{a_n}$, for $n > m$, and $b_n = 1$ for $n \leq m$.

The real number $\eta = [b_n]$ has the desired property.

The real number η is called the *multiplicative inverse* of ξ and is denoted by ξ^{-1}.

(*vi*) The multiplicative inverse ξ^{-1} is unique.

Theorem 6. *Prove that* $(R, +, .)$ *is a field.*
The proof is left to the reader.

5. ORDER IN R

We shall now give an *order structure* to the field of real numbers. To do so we shall first define the set of positive elements of **R**.

5.1 Definition (*A positive sequence of rational numbers*): *A rational sequence* $\{a_n\}$ *is called a positive sequence if there exists a positive rational number e and a positive integer m, such that*

$$a_n > e, \text{ for all } n \geq m$$

From the above definition it may be easily shown that *if* $\{a_n\}$ *and* $\{b_n\}$ *are positive sequences of rational numbers, then* $\{a_n + b_n\}$ *and* $\{a_n b_n\}$ *are also positive sequences of rational numbers.*

Theorem 7. *If* $\{a_n\} \sim \{a_n'\}$, *and* $\{a_n\}$ *is a positive rational sequence, then* $\{a_n'\}$ *is also a positive rational sequence.*

Since $\{a_n\}$ is a positive rational sequence, therefore, $\exists\, e > 0$ in $\mathbf{Q}^+$ and $m_1 \in \mathbf{N}$, such that

$$a_n > e, \text{ for } n \geq m_1 \qquad \ldots(1)$$

Again, since $\{a_n\} \sim \{a_n'\}$, therefore $\exists\, m_2 \in \mathbf{N}$, such that

$$|\, a_n' - a_n \,| < \frac{e}{2}, \text{ for all } n \geq m_2$$

or

$$-\frac{e}{2} < a_n' - a_n < \frac{e}{2}, \text{ for } n \geq m_2 \qquad \ldots(2)$$

Let $m_0 = \max(m_1, m_2)$.

$\therefore$ $$a_n' = (a_n' - a_n) + a_n$$

$$> -\frac{e}{2} + e, \text{ for } n \geq m_0 \qquad \text{[from (1) and (2)]}$$

$$= \frac{e}{2} > 0, \text{ for all } n \geq m_0$$

Hence $\{a_n'\}$ is a positive rational sequence.

Corollary. If ξ is a real number and $\{a_n\} \in \xi$ be a positive sequence in $\mathbf{Q}$ then every sequence $\{a_n'\} \in \xi$ is also a positive sequence in $\mathbf{Q}$.

5.2 Definition. *A real number ξ is positive if every sequence in ξ is a positive sequence.*

In view of Theorem 7, it follows that *a real number ξ is positive if and only if there exists a positive rational sequence in ξ.*

We shall denote the set of positive real numbers by $\mathbf{R}^+$. Thus

$$\mathbf{R}^+ = \begin{cases} \{\xi \in \mathbf{R} \mid \xi \text{ is positive}\} \\ \{\xi \in \mathbf{R} \mid \text{for some } \{a_n\} \in \xi, \{a_n\} \text{ is positive}\} \end{cases}$$

From definition, the following result may be easily proved.

If ξ, η *are positive real numbers then so also are* $\xi + \eta$ and $\xi \cdot \eta$.

Theorem 8. *If $\xi \in \mathbf{R}$, then one and only one of the following statements is true*:

(*i*) $\xi = 0$, (*ii*) $\xi \in \mathbf{R}^+$, (*iii*) $-\xi \in \mathbf{R}^+$.

We first show that at *least one* of the three statements is true.

Let $\xi = [a_n]$, so that $\{a_n\} \in \xi$, and $\{a_n\} \in F_Q$.

When $\xi = 0$, there is nothing to prove, for then (*i*) holds.

Let $\xi \neq 0$, so that $\{a_n\}$ is not equivalent to $\{0_n\}$, where $0_n = 0$ for all n, consequently $\{a_n\}$ does not converge to 0. Hence, $\exists\, e \in \mathbf{Q}^+$ and $m_1 \in \mathbf{N}$, such that

$$|a_n| > e \quad \forall\ n \geq m_1 \qquad ...(1)$$

Again, since $\{a_n\}$ is a Cauchy sequence and $e > 0$, therefore $\exists\, m_2 \in \mathbf{N}$, such that

$$|a_{n+p} - a_n| < \frac{e}{2}, \text{ for } n \geq m_2,\ p \geq 1 \qquad ...(2)$$

Let $m = \max(m_1, m_2)$, then from (2),

$$a_m - \frac{e}{2} < a_{m+p} < a_m + \frac{e}{2},\ p \geq 1 \qquad ...(3)$$

From (1) either $a_m > e$ or $a_m < -e$.

If $a_m > e$, then from (3),

$$a_{m+p} > e - \frac{e}{2} = \frac{e}{2} > 0,\ p \geq 1$$

Therefore, $\{a_n\}$ is a positive sequence and hence $\xi \in \mathbf{R}^+$.

And if $a_m < -e$, then again from (3),

$$a_{m+p} < -e + \frac{e}{2} = -\frac{e}{2},\ p \geq 1$$

i.e., $$-a_{m+p} > \frac{e}{2} > 0,\ p \geq 1$$

Therefore $\{-a_n\}$ is a positive sequence and hence $-\xi \in \mathbf{R}^+$.

Thus we have shown that at least one of the three statements is true.

We now proceed to show that *not more than one* of the three statements is true.

If $\xi = 0$, then $\{a_n\} \sim \{0_n\}$. Hence for rational $e > 0, \exists\, m' \in \mathbf{N}$ such that

$$|a_n| < e, \text{ for } n \geq m'$$

Hence, there is no $e \in \mathbf{Q}^+$ such that for some $m_0 \in \mathbf{N}$, either

$$a_n \geq e \quad \forall\ n \geq m_0$$

or

$$-a_n \geq e \quad \forall\ n \geq m_0$$

Thus if $\xi = 0$, then neither $\xi \in \mathbf{R}^+$ nor $-\xi \in \mathbf{R}^+$.

Now, if possible, $\xi \in \mathbf{R}^+$ and $-\xi \in \mathbf{R}^+$.

Then for some e, e' ($e < e'$, say) $\in \mathbf{Q}^+$ and $n_1, n_2 \in \mathbf{N}$,

$$a_n \geq e \quad \forall\ n \geq n_1$$
$$-a_n \geq e' \quad \forall\ n \geq n_2$$

Hence, for $n = \max(n_1, n_2)$,

$$0 < e' \leq -a_n \leq -e < 0$$

which is impossible,

$\therefore$ $$\xi \in \mathbf{R}^+ \ \wedge\ -\xi \in \mathbf{R}^+ \text{ is false}$$

Hence the theorem.

5.3 *Thus* $\mathbf{R}^+$ *is a set of positive elements of* $\mathbf{R}$ *such that*

(*a*) *If* $\xi, \eta \in \mathbf{R}^+$, *then* $\xi + \eta \in \mathbf{R}^+$ *and* $\xi.\eta \in \mathbf{R}^+$.

(*b*) *For each* $\xi \in \mathbf{R}$, *one and only one of the following is true*:

(*i*) $\xi \in \mathbf{R}^+$ (*ii*) $\xi = 0$ (*iii*) $-\xi \in \mathbf{R}^+$.

5.4 **Definition.** *A real number* ξ *is said to be greater than a real number* η if $\xi - \eta \in \mathbf{R}^+$.

Using the symbol > to denote '*greater than*', we write

$$\xi > \eta \text{ iff } \xi - \eta \in \mathbf{R}^+$$

The same thing can also be expressed by saying that η is '*less than*' ξ and write $\eta < \xi$.

EXERCISE

1. For real numbers $\xi, \eta, \zeta, \ldots$, prove that
   (1) Exactly one of the following holds:
   (*i*) $\xi > \eta$, (*ii*) $\xi = \eta$, (*iii*) $\xi < \eta$.
   (2) $\xi > \eta$ and $\eta > \zeta \Rightarrow \xi > \zeta$.
   (3) $\xi > \eta \Rightarrow \xi + \zeta > \eta + \zeta$.
   (4) For $\zeta \in \mathbf{R}^+$, $\xi > \eta \Rightarrow \xi \cdot \zeta > \eta \cdot \zeta$.

Theorem 9. *Prove that* $(\mathbf{R}, +, \cdot, >)$ *is an ordered field.*

The proof is left to the reader.

6. REAL RATIONAL AND IRRATIONAL NUMBERS

If a Cauchy sequence of rational numbers converges in the field of rationals then the real number determinded by it is called a *real rational number*. Thus, if a rational Cauchy sequence $\{a_n\}$ converges to a rational number α then $[a_n]$ is the real rational number α.

Theorem 10. *If* α *is a rational number then there exists a rational Cauchy sequence converging to* α.

Let $\{a_n\}$, where $a_n = \alpha$ for all n, be a rational sequence, which clearly converges to α. Also it is a Cauchy sequence, since $|a_{n+p} - a_n| = 0$ for all n, p.

Theorem 11. *If* $\{a_n\}$ *and* $\{b_n\}$ *are two rational Cauchy sequences converging to the same limit* α, *then*

$$\{a_n\} \sim \{b_n\}.$$

$\because$ $\lim a_n = \alpha = \lim b_n$

$\therefore$ $\lim (a_n - b_n) = 0$

$\therefore$ $\{a_n\} \sim \{b_n\}$

Theorem 12. *To every rational number* α *there corresponds a unique real rational number.*

Let $\{a_n\}$ be a rational Cauchy sequence that converges to α, so that $\{a_n\}$ is a real rational number that corresponds to α.

If $[a_n']$ be another real rational number that corresponds to α, then the rational Cauchy sequence $\{a_n'\}$ also converges to α. Hence, by Theorem 11,

$$\{a_n\} \sim \{a_n'\}$$

$\therefore$ $[a_n] = [a_n']$.

Notation. The set of all real rational numbers will be denoted by $\mathbf{R}^*$, where $\mathbf{R}^* = \{\alpha \mid \alpha$ is a real rational number$\}$.

If $\xi \in \mathbf{R}$ but $\xi \notin \mathbf{R}^*$, then ξ is called *real irrational* number and the set of all real irrational numbers is $\mathbf{R} - \mathbf{R}^*$.

7. SOME PROPERTIES OF REAL NUMBERS

Theorem 13. *If* ξ *is a real number and* $\{x_n\}$ *a Cauchy sequence in* **Q** *such that* $\{x_n\} \in \xi$, *then* $\lim x_n = \xi$ *in* **R**.

Let $\varepsilon > 0$ be a real number and e a rational number such that $0 < e < \varepsilon$.

Since $\{x_n\}$ is a Cauchy sequence in **Q**, therefore for $e > 0, \exists\, m_0 \in \mathbf{N}$ such that

$$|x_n - x_m| < \frac{e}{2} \quad \forall\ m, n \geq m_0$$

Hence for all $m, n \geq m_0$,

$$e - |x_n - x_m| > \frac{e}{2}$$

and so, for each $n \geq m_0$ the sequence $\{y_m\}$, where $y_m = e - |x_n - x_m|$, is a positive sequence in **R**.

$$\therefore \qquad [y_m] > 0 \text{ in } \mathbf{R}$$

or

$$e - [|x_n - x_m|] > 0 \text{ in } \mathbf{R}$$

Thus for $n \geq m_0$

$$\begin{aligned} |x_n - \xi| &= |x_n - [x_m]| \\ &= |[x_n] - [x_m]| \\ &= |[x_n - x_m]| = [|x_n - x_m|] < e < \varepsilon \end{aligned}$$

$$\therefore \qquad \lim x_n = \xi, \text{ and } \xi \in \mathbf{R}.$$

Corollary 1. If $\xi \in \mathbf{R}$ *and* $\varepsilon > 0$ in **R** then there is an $x \in \mathbf{Q}$ such that $|\xi - x| < \varepsilon$ in **R**.

Let $\{x_n\}$ be a Cauchy sequence in **Q** such that $\{x_n\} \in \xi$.

$$\therefore \qquad \lim x_n = \xi \text{ in } \mathbf{R}$$

Hence for every $\varepsilon > 0$ in **R**, $\exists$ an $m \in \mathbf{N}$ such that

$$|\xi - x_n| < \varepsilon \text{ in } \mathbf{R} \quad \forall\ n \geq m$$

In particular for $x = x_m \in \mathbf{Q}$,

$$|\xi - x| < \varepsilon$$

Corollary 2. If $\xi < \eta$ in **R**, then there is an $x \in \mathbf{Q}$ such that $\xi < x < \eta$.

We know that every ordered field is dense. Therefore, there is a real number ζ such that $\xi < \zeta < \eta$. If $\varepsilon = \min(\zeta - \xi, \eta - \zeta)$, then by Corollary 1. There is a rational number x such that

$$\xi < \zeta - \varepsilon < x < \zeta + \varepsilon < \eta.$$

Corollary 3. **R** is Archimedean

or

For each pair of positive real numbers ξ, η, there exists a positive integer n such that $n\ \xi > \eta$.

For $0 < \xi < \eta$ in **R**, let x, y be rational numbers such that $0 < x < \xi < \eta < y$ in **R**. Since the field **Q** is Archimedean, therefore $\exists$ an $n \in \mathbf{N}$ such that $nx > y$.

$\therefore \quad n\xi > nx > y > \eta$

Hence **R** is Archimedean.

Theorem 14. *Every Cauchy sequence of Cantor real numbers converges in* **R**.

Let $\{\xi_n\}$ be a Cauchy sequence in **R**. Then for $\varepsilon > 0$ in **R**, $\exists\, m_1 \in \mathbf{N}$ such that

$$|\xi_n - \xi_m| < \varepsilon/3 \qquad \forall\ m, n \geq m_1 \qquad ...(1)$$

By Corollary 1, for each $n \in \mathbf{N}$, there exists a rational number x_n such that $|\xi_n - x_n| < 1/n$.

For above $\varepsilon > 0$, we can choose $m_2 \in \mathbf{N}$ such that

$$1/n < \frac{1}{3}\varepsilon \text{ and } |\xi_n - x_n| < 1/n < \frac{1}{3}\varepsilon \qquad \forall\ n \geq m_2 \qquad ...(2)$$

Let $m_0 = \max(m_1, m_2)$, from (1) and (2),

$$\begin{aligned} |x_n - x_m| &= |x_n - \xi_n + \xi_n - \xi_m + \xi_m - x_m| \\ &\leq |x_n - \xi_n| + |\xi_n - \xi_m| + |\xi_m - x_m| \\ &< \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon, \quad \forall\ m, n \geq m_0 \end{aligned}$$

$\therefore$ $\{x_n\}$ is a Cauchy sequence in **Q**.

Hence $[x_n]$ is a real number ξ, say,

$\therefore$ $\lim x_n = \xi$ in **R** [Theorem 13]

Again, since $\{x_n\}$ converges to ξ in **R**, for $\varepsilon > 0$ in **R**, $\exists$ an $m_3 \in \mathbf{N}$ such that

$$|x_n - \xi| < \frac{2}{3}\varepsilon, \qquad \forall\ n \geq m_3 \qquad ...(3)$$

Hence for $n \geq m' = \max(m_2, m_3)$, we have

$$\begin{aligned} |\xi_n - \xi| &= |\xi_n - x_n + x_n - \xi| \\ &\leq |\xi_n - x_n| + |x_n - \xi| \\ &< \frac{1}{3}\varepsilon + \frac{2}{3}\varepsilon = \varepsilon \text{ [using (2), (3)]} \end{aligned}$$

$\therefore \quad \lim \xi_n = \xi$ in **R**

8. COMPLETENESS IN R

Theorem 15. (***Order-completeness property***). *Every non-empty subset of real numbers which is bounded above has the supremum in* **R**.

(*i*) Suppose A is a non-empty subset of **R**, b is an upper bound of A, and $\alpha \in A$ so that $\alpha \leq b$.

Since **R** is Archimedean, therefore, for each $n \in \mathbf{N}$, there exists an $\overline{m} \in \mathbf{N}$ such that $\alpha + (\overline{m}/n) \geq b$ in A, and therefore $\alpha + (\overline{m}/n)$ is an upper bound of A. Hence for each $n \in \mathbf{N}$, the set

$$B_n = \left\{ m \,|\, \alpha + \frac{m}{n} \text{ is an upper bound of } A, m \in \mathbf{N} \right\}$$

is a non-empty subset of **N**.

But, since every non-empty subset of Natural numbers has a least element, let m_n be the least element of B_n. Thus for each $n \in \mathbf{N}$,

(1) $y_n = \alpha + \dfrac{m_n}{n}$ is an upper bound of A, and

(2) $x_n = y_n - \dfrac{1}{n} = \alpha + \dfrac{m_n - 1}{n}$ is not an upper bound of A.

$\therefore \qquad x_m < y_n, \quad \forall\ m, n \in \mathbf{N}$

Now

$$x_n - x_m < y_m - x_m = \frac{1}{m}$$

and

$$x_m - x_n < y_n - x_n = \frac{1}{n}$$

$\therefore \qquad |x_n - x_m| = \max\ (x_n - x_m, x_m - x_n)$

$$< \max\left(\frac{1}{m}, \frac{1}{n}\right), \quad \forall\ m, n \in \mathbf{N}$$

$$< \frac{1}{n_0}, \quad \text{for } m, n \geq n_0$$

Hence $\{x_n\}$ is a Cauchy sequence in $\mathbf{R}$ which by Cauchy property of $\mathbf{R}$ (Theorem 14, § 7) converges in $\mathbf{R}$.

$\therefore \qquad \lim x_n = \xi$, where $\xi \in \mathbf{R}$

(*ii*) We shall now show that $\xi = \sup A$.

Let, if possible, ξ be not an upper bound of A.

Hence, $\xi < x$, for some x in A.

Since $\lim x_n = \xi$ and $\lim 1/n = 0$, there is some $n \in \mathbf{N}$, such that

$$\frac{1}{n} < \frac{x - \xi}{2}$$

and

$$x_n - \xi \leq |x_n - \xi| < \frac{x - \xi}{2}$$

Then by (1) and (2), we have

$$y_n = x_n + \frac{1}{n} < \left(\xi + \frac{x - \xi}{2}\right) + \frac{x - \xi}{2} = x \text{ in } \mathbf{R}$$

But this is impossible by (1), since $x \in A$.

Hence ξ is an upper bound of A.

If ξ is not the least upper bound, let $\eta < \xi$ be the least upper bound of A.

Let $\xi - \eta = \delta > 0$.

Since $\lim x_n = \xi$, therefore for $\delta > 0, \exists\, n \in \mathbf{N}$ such that

$$\xi - x_n \leq |\xi - x_n| < \delta = \xi - \eta$$

$\therefore$ $$\eta < x_n \leq x, \text{ for some } x \in A$$

which is a contradiction.

Hence $\xi \leq \eta$, so that ξ is the least upper bound of A.

Thus, the non-empty set A of real numbers has the supremum in $\mathbf{R}$.

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